On Euler systems and Nekovář–Selmer complexes

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We develop a theory of Euler and Kolyvagin systems relative to the Nekovář–Selmer complexes of p-adic representations over local complete Gorenstein rings. This is simultaneously finer, and requires weaker hypotheses, than the theory of Kolyvagin systems developed by Mazur and Rubin over discrete valuation rings and then by Sakamoto, Sano and the second author over Gorenstein rings. To illustrate its advantages, we prove new cases of Kato's generalised Iwasawa main conjecture for $\mathbb{Z}_p(1)$ and the p-adic Tate modules of rational elliptic curves, as well as of the Quillen–Lichtenbaum conjecture, and we also strengthen existing results on the Birch–Swinnerton-Dyer conjecture for CM elliptic curves.

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1. Introduction

Background and main results The notion of 'Euler systems' was introduced by Kolyvagin in [65] as a means of axiomatising parallel earlier work of his on the Selmer groups of modular elliptic curves [64] and of Thaine on the class groups of real abelian fields [105]. The general theory was then further developed by Rubin [92], by Kato [59], by Perrin-Riou [87] and by Mazur and Rubin [74, 75], and has by now become well-established as an effective method for studying the Selmer groups of p-adic Galois representations. In particular, it has led to spectacular advances concerning the Birch and Swinnerton-Dyer conjecture and other important cases of the 'Tamagawa number conjecture' of Bloch and Kato [5] concerning motivic L-functions – for examples, see [92] and [60] and, more recently, [68], [63], [70], and [97].

In this article, we focus attention on the class of 'refined' special value conjectures for motives with extra symmetries. We recall these conjectures range from very concrete examples such as the 'refined conjecture of BSD-type' of Mazur and Tate [77] for the modular symbols of rational elliptic curves to the unifying, and highly abstract, 'generalised Iwasawa main conjecture' of Kato [57, 58] (or, equivalently, the commutative-coefficient case of the subsequently formulated 'equivariant Tamagawa number conjecture' [18]) that is stated in terms of the Knudsen–Mumford determinants of complexes arising from the p-adic realisations of motives. We further recall that, even granting the existence of an appropriate Euler system, evidence for these refined conjectures remains, in general, both rather limited and also often conditional on difficult-to-verify hypotheses such as the vanishing of Iwasawa μ-invariants or order-of-vanishing conditions on p-adic L-functions.

With these things in mind, the main aim of the present article is to render Euler systems directly applicable to the study of refined special value conjectures by systematically incorporating complexes into the foundations of the theory. In the course of doing so, we shall also be able to weaken some of the hypotheses that have become standard in this area.

Broadly speaking, the main achievement of this article will therefore be the development of a working theory of Euler systems that are valued in (exterior power biduals of) the cohomology groups of the Selmer complexes introduced by Nekovář in [80] rather than in Galois cohomology groups themselves. Unsurprisingly, the idea of incorporating Selmer complexes into the theory of Euler systems originates with Nekovář himself, with the explicit question of [80, § 0.19.3] being motivated by the observation that formulating Iwasawa main conjectures in the setting of Selmer complexes, as pioneered in [57], can explain the trivial zeros of p-adic L-functions. There are, however, two key obstacles that need to be overcome in order for this to be properly achieved in our context.

Firstly, previous approaches rely heavily on the assumption that the Kolyvagin systems being considered do not become trivial upon reduction to the residual representation. For representations over discrete valuation rings, a conjecture of Kolyvagin suggests that this is indeed a mild restriction for the Kolyvagin systems related (via an Euler system) to L-values (cf. [28] and the references therein). However, in our more general setting the presence of 'trivial zeros mod p' will often force residual triviality even for Kolyvagin systems arising from L-series, and so we must overcome the associated technical difficulties. At the same time, this possibility of residual triviality also means that the 'core rank' of Mazur and Rubin of a given Selmer structure can be strictly negative, and in any such case one cannot expect the existence of the 'core vertices' that are pivotal to their approach. To resolve this problem, we combine the Cebotarev density theorem with the Artin–Verdier duality theorem to prove the existence of a weaker, relative, version of core vertex that we refer to as a 'relative core vertex'.

Secondly, when working over a general Gorenstein ring the relevant Selmer groups may not be free at a relative core vertex or even, under our weaker hypotheses, at a core vertex (should one exist) and this notably complicates the necessary analysis. To deal with this problem, we are therefore forced to keep careful track of certain 'error terms' that occur naturally when one attempts to extend the recent work of Sakamoto, Sano and the second author [26, 25],

or the related work of Kataoka [55] and Kataoka–Sano [56], to the setting of Nekovář–Selmer complexes, with the ultimate aim of providing 'bounds' for these error terms in the limit.
Whilst the main advances in the general theory that we obtain are therefore somewhat technical in nature (see, for example, Theorems 4.20, 5.27 and 6.38), they have significant advantages over previous results in this area. In the first instance, they are finer since their conclusions concern the determinants of Selmer complexes rather than either the Fitting or characteristic ideals of Selmer groups, thereby directly allowing us, for example, to remove the '$\mu = 0$' hypotheses from arguments related to Kato's conjectures. In addition, they are broader in that they consider Euler systems relative to a general class of Nekovář structures and hence, for example, incorporate arguments such as those of Kings–Loeffler–Zerbes in [63, § 12] concerning Kolyvagin systems with Greenberg local conditions. Moreover, our theory is developed for general local complete Gorenstein rings with finite residue fields of characteristic p, and so is applicable to the study of Kato's conjecture in both the 'Galois-equivariant case' discussed below and also relative to deformation rings, as studied by Fouquet in [42, 43].
For all of these reasons, our results can be combined with known techniques to give, even in classical settings, stronger versions of well-known results.

Selected arithmetic consequences To illustrate the latter point concretely, we shall first use our approach in the setting of Kato's Euler system of zeta elements to prove results such as the following. This result is a direct consequence of (the more general) Corollary 9.6 and Remark 9.8 and, before stating it, we recall that Kato's generalised Iwasawa main conjecture asserts an equality of lattices and so is equivalent to the validity of two inclusions.

Theorem. *Let E be a rational elliptic curve and K a finite abelian extension of $\mathbb{Q}$ for which the Hasse–Weil L-function $L(E/K, s)$ does not vanish at $s = 1$. Fix a prime $p > 3$ such that*

- *the action of $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$ on the p-adic Tate module of E contains $\mathrm{SL}_2(\mathbb{Z}_p)$, and*
- *either K contains no root of unity of order p or E has potentially good reduction at p.*

Then the pair $(h^1(E/K)(1), \mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})])$ validates one inclusion in Kato's generalised Iwasawa main conjecture. Further, this case of the conjecture is valid in full if, in addition, $K/\mathbb{Q}$ is a p-extension and E has good ordinary reduction at p.

We note that this result, in particular, avoids any hypotheses relating to 'trivial zeros mod p' (such as p being 'non-anomalous' in the terminology of Mazur [72]) that have been used in previous work in this area (see, for example, the discussion in [25, § 6.4.1]). For this reason, it leads to concrete new evidence in support of the precise conjecture of Birch and Swinnerton-Dyer (see Corollary 9.9). In addition, in the companion article [14] of Honnor and the first author, the above result plays a pivotal role in the verification of a large part of the conjectures of Mazur and Tate [77] on modular symbols mentioned earlier.
As a further application, we consider the multiplicative group over a number field k. In this setting, our approach leads to evidence for Kato's conjecture that is conditional only on a conjectural integrality property of the Rubin–Stark Euler system (see Theorems 10.3 and 10.14). In particular, if k is an imaginary quadratic field, then the latter system is the (integral) Euler system of elliptic units and this allows us to prove the following unconditional result (which follows immediately from Theorem 10.4 and Corollary 10.5).

Theorem. *Let K be a finitely abelian extension of an imaginary quadratic field k. Then for every prime $p > 3$ and integer $j \geq 1$, Kato's generalised Iwasawa main conjecture is valid for $(h^0(\mathrm{Spec}\, K)(1-j), \mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})])$.*

This theorem completes earlier partial results of Bley [3, 4], of Johnston-Leung and Kings [53], and of Hofer and the first author [13]. It also implies a variety of more explicit conjectures for abelian extensions of imaginary quadratic fields, ranging from the 'Rubin–Stark conjecture' from [91] to the 'refined class number formula' conjecture of Mazur–Rubin [76] and Sano [99] and several conjectures in classical Galois module theory (see [17] for more details regarding these connections). At the same time, the above result leads to the verification of the Quillen–Lichtenbaum conjecture in a new family of cases (see Remark 10.8) and combines with the general strategy described by Kato in [58, Ch. I, § 3.3] (see also [11]) to obtain the following 'equivariant' refinement of the main result of Burungale and Flach [30] relating to the Birch and Swinnerton-Dyer conjecture for CM elliptic curves (see Corollary 10.12).

Theorem. *Let E be an elliptic curve defined over a number field F containing an imaginary quadratic field k. Assume E has complex multiplication by the ring of integers of k and $F(E_{\mathrm{tors}})/k$ is abelian, and write B for the Weil restriction of E to k. Then, for any finite abelian extension K of k with $L(B/K, 1) \neq 0$, and every prime $p > 3$, Kato's generalised Iwasawa main conjecture is valid for the pair $(h^1(B/K)(1), (\mathrm{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Z}_p)[\mathrm{Gal}(K/k)])$.*

At this point, it seems worth remarking that, whilst the above applications involve classical (rank-one) Euler systems, their proofs still involve a refined version of the higher-rank theory of Stark systems introduced by Mazur and Rubin and we are not aware of an approach that would avoid these higher–rank aspects. For example, the (rank-one) proof strategy used by Rubin [92] or Mazur–Rubin [74] relies in an essential way on algebraic results such as the structure theorem for Iwasawa modules that are not available for the more general coefficient rings that we must work with.

Finally we remark that, as mentioned earlier, Euler systems related to special values of L-series are now known to exist in a variety of other important settings and in each of these cases our theory will have applications. We aim to return to this in future work.

Overview of contents For clarity of exposition, we have divided this article in two parts. In Part I, we extend previous work of Mazur and Rubin [74, 75] and of Sakamoto et al. [26, 25] in order to develop a general theory of Euler, Kolyvagin and Stark systems for Nekovář–Selmer complexes relative to a morphism $\mathcal{R} \to R$ of local complete Gorenstein rings. In Part II (comprising § 8 to § 10), we then illustrate this general theory by presenting some concrete arithmetic applications, including those discussed above. In fact, a reader who is mainly interested to know how our techniques apply to arithmetic problems may prefer to simply read § 4, for a discussion of key concepts and hypotheses and the statement of Theorem 4.20, and then directly pass to Part II.

In a little more detail, then, the main contents of this article is as follows. In § 2 we review the families of rings that arise in the development of our theory and then establish necessary technical results concerning Matlis duals, exterior biduals, Fitting ideals, and resolutions and determinants of complexes. In § 3 we review Selmer structures both in the sense of Nekovář, which we refer to as Nekovář(–Selmer) structures, and of Mazur and Rubin, which we refer to as Mazur–Rubin(–Selmer) structures, and establish some useful relations between the cohomology groups of the Selmer complex of a Nekovář structure and the Selmer groups of an associated Mazur–Rubin structure and its dual. We also study dual Nekovář structures, describe relations between Selmer complexes that follow from the Artin–Verdier duality theorem, and construct a family of perfect Selmer complexes that plays a vital role in our theory. In § 4, we introduce a notion of Euler systems relative to Nekovář structures, discuss the hypotheses under which our theory can be developed and then state our main technical result (Theorem 4.20) concerning relations between the values of an Euler system for a given Nekovář structure and the determinant of its associated Selmer complex. The following two sections then comprise the technical heart of our article as we extend previous arguments relating to Kolyvagin systems for p-adic representations over finite self-injective rings. Firstly, in § 5, as a replacement for the 'modified Selmer structures' of Mazur and Rubin that play a key role in the theory of [26, 25] we study the Selmer complexes of a natural family of 'modified Nekovář structures', and also prove the existence of a 'Kolyvagin derivative homomorphism' in the setting of Euler and Kolyvagin systems relative to Nekovář structures. Then, in § 6, we prove the existence in our setting of relative core vertices and investigate the extent to which they can be used to control the value at the unit modulus of a Kolyvagin system relative to a Nekovář structure. In § 7 we combine the main results of § 5 and § 6 with an algebraic construction of Stark systems and a delicate limit argument to prove Theorem 4.20. Partly with future applications in mind, in an appendix to Part I we also present an axiomatic treatment of the theory of algebraic Stark systems. Then, as a first step towards arithmetic applications, in § 8 we explain how to apply Theorem 4.20 for certain 'modified, relaxed' Nekovář structures in order to study the general case of Kato's generalised Iwasawa main conjecture. Finally, in § 9 and § 10 we present applications of this strategy and, in particular, prove all results stated in the introduction by relying on the Euler systems of Kato's zeta elements and elliptic units.

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Part I. The general theory

In this first part of the article we develop a general theory of Euler and Kolyvagin systems for Nekovář–Selmer complexes over local complete Gorenstein rings. Arithmetic consequences of this theory will be discussed in the second part of the article (starting in § 8).

2. Algebraic preliminaries

This section establishes some general algebraic results that are essential to theory developed in later sections. Throughout, we fix a commutative Noetherian ring R. For each natural number n, we write $\mathrm{Spec}^n(R)$ and $\mathrm{Spec}^{\le n}(R)$ for the sets of prime ideals of R that are of height n and of height at most n respectively.
We endow the linear dual $M^* \coloneqq \mathrm{Hom}_R(M, R)$ of an R-module M with the structure of an R-module by setting $(x \cdot f)(m) = x \cdot f(m)$.
We frequently use (without explicit comment) that inverse limits over systems indexed by the natural numbers are exact on the category of finitely generated $\mathbb{Z}_p$-modules.
Throughout the article we also use the following convenient notations: we set $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$ and, for $n \in \mathbb{N}$, write $[n]$ for the ordered set $\{i \in \mathbb{N} : i \le n\}$.

2.1. Hypotheses on rings

2.1.1. G_n-rings

We first introduce the categories of rings for which most of our theory will be developed.

(2.1) Definition. *Let R be a commutative Noetherian ring. For $n \in \mathbb{N}_0$ one says that*

- *R has property (G_n) if $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Spec}^{\le n}(R)$;*
- *R has property (S_n) if $\mathrm{depth}(R_\mathfrak{p}) \ge \inf(n, \mathrm{ht}(\mathfrak{p}))$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$;*
- *R is a 'G_n-ring' if it has both of the properties (G_{n-1}) and (S_n).*

(2.2) Remark. (i) Condition (S_n) is often referred to as 'Serre's condition'.

(ii) G_n-rings were first studied by Ischebeck [51] and Reiten and Fossum [89]. G_2-rings are also studied by Vasconcelos [106, 107] who refers to them as 'quasi-normal rings'.

(iii) A ring is Cohen–Macaulay if and only if it satisfies (S_n) for all $n \in \mathbb{N}_0$, and Gorenstein if it satisfies (G_n) for all $n \in \mathbb{N}_0$. Since every Gorenstein ring is Cohen–Macaulay, this also shows that Gorenstein rings satisfy both (S_n) and (G_n) for all $n \in \mathbb{N}_0$.

Our first result shows that the class of G_n-rings is closed under a variety of natural operations. In particular, it implies (equivariant) Iwasawa algebras and group rings of finite abelian groups with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_p$ are G_n-rings (in fact, Gorenstein rings).

(2.3) Lemma. *If R is a G_n-ring for some $n \in \mathbb{N}_0$, then the following claims are valid:*

(i) The polynomial ring $R[t]$ for an indeterminate t is a G_n-ring.

(ii) The ring of formal power series $R[\![t]\!]$ is a G_n-ring.

(iii) The group ring $R[G]$ for a finite abelian group G is a G_n-ring.

Proof. Claim (i) is proved in [89, Cor. (a) after Prop. 1] and (ii) is [89, Cor. after Prop. 3]. To prove (iii) we apply [89, Prop. 1 (ii)]: to see that the fibres of $R \to R[G]$ are n-Gorenstein it suffices to note that, for any field F, the ring $F[G]$ is Gorenstein (this follows from [34, Cor. 8] which shows that $\mathrm{injdim}_{F[G]}(F[G]) = \mathrm{injdim}_F(F) = 0$). □

An R-module M is said to be 'pseudo-null' if $M_\mathfrak{p}$ vanishes for all $\mathfrak{p} \in \mathrm{Spec}^{\leq 1}(R)$. The following observations about this notion will be useful.

(2.4) Lemma (Sakamoto). *Fix an S_2-ring R and an R-module M.*

(i) If M is pseudo-null, then M^* and $\mathrm{Ext}^1_R(M,R)$ both vanish.

(ii) If R is also a G_0-ring, and N_1 and N_2 are finitely generated submodules of M with $(N_1)_\mathfrak{p} \subseteq (N_2)_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}^{\leq 1}(R)$, then the natural map $(N_1)^{**} \to (N_2)^{**}$ is injective.

Proof. Claim (i) is proved in [95, Lem. B.11] and (ii) follows from the proof of [95, Lem. C.13]. In fact, since (ii) is slightly more general than the statement given in *loc. cit.*, we reproduce the argument here. By assumption, the module $(N_1 + N_2)/N_2$ is pseudo-null and so, by (i), the natural map $(N_1+N_2)^* \to N_2^*$ is bijective. Hence, by [95, Lem. C.1] (with $r = 1$), the natural map $(N_1)^{**} \to (N_1+N_2)^{**} \cong (N_2)^{**}$ is injective, as claimed. □

2.1.2. Complete Gorenstein rings

In this section we fix a prime number p and Noetherian ring R that satisfy the following condition.

$$R \text{ is a local complete Gorenstein ring with finite residue field of characteristic } p. \quad (2.5)$$

We write d for the dimension of R, $\mathfrak{m}$ for its maximal ideal and $\Bbbk$ for its residue field $R/\mathfrak{m}$. We also fix a decreasing filtration

$$\mathfrak{a}_0 := \mathfrak{m} \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \dots \quad (2.6)$$

of ideals of R such that, for every n, the ring $R_n := R/\mathfrak{a}_n$ is Gorenstein of dimension zero (or equivalently, finite and self-injective) and the natural map $R \to \varprojlim_n (R/\mathfrak{a}_n)$ is bijective.

(2.7) Remark. The condition (2.5) ensures that R is Cohen-Macauley and so one can fix a regular sequence $\{x_i\}_{i\in[d]}$ in $\mathfrak{m}$ with $\sqrt{(x_1,\dots,x_d)} = \mathfrak{m}$. One then obtains a filtration (2.6) of the required sort by setting $\mathfrak{a}_n := (x_1^n,\dots,x_d^n)$ for all $n \geq 1$. Indeed, $\{x_i^n\}_{i\in[d]}$ is a regular sequence, $\sqrt{\mathfrak{a}_n} = \mathfrak{m}$, the quotient $R/\mathfrak{a}_n$ is Gorenstein (cf. [7, Prop. 3.1.19 (b)]) of dimension zero and the map $R \to \varprojlim_n R/\mathfrak{a}_n$ is bijective since R is complete, Noetherian, and local. Further, this filtration is universal in the sense that if $\{\mathfrak{a}'_n\}_{n\in\mathbb{N}}$ is any other filtration as in (2.6), then for every n there exists $m \in \mathbb{N}$ with $\mathfrak{a}_m \subseteq \mathfrak{a}'_n$. (This is true since $R/\mathfrak{a}'_n$ being zero-dimensional implies $R/\mathfrak{p}$ is zero-dimensional for any associated prime $\mathfrak{p} \subseteq R$ of $R/\mathfrak{a}'_n$ and hence that $R/\mathfrak{p}$ is field; it follows that $\mathfrak{m}$ is the only associated prime of $R/\mathfrak{a}'_n$ and hence that $\mathfrak{a}'_n$ is $\mathfrak{m}$-primary, as required.) The following special instances of the filtration (2.6) will also be of importance to us.

(i) R is finite, local and self-injective: for every $n \geq 1$ one can set $\mathfrak{a}_n = (0)$ so $R_n = R$.

(ii) R is a $\mathbb{Z}_p$-order that is local and Gorenstein: one can set $\mathfrak{a}_n = (p^n)$ for $n \geq 1$, so $R_n = R/p^nR$.

(iii) $d > 1$ and R is the completed group ring $\mathbb{Z}_p[\![G]\!]$ of an abelian pro-p compact p-adic Lie group G of rank $d-1$. Fix a base $\{U_n\}_{n\geq 1}$ of open neighbourhoods of the identity of G with $U_{n+1} \subset U_n$ for all n. Then, for each $n \geq 1$, one can take $\mathfrak{a}_n$ to be the kernel of the canonical (surjective) projection map from $R = \mathbb{Z}_p[\![G]\!]$ to $R_n := (\mathbb{Z}_p/p^n\mathbb{Z}_p)[G/U_n]$.

We fix an injective envelope $E_R(\Bbbk)$ of the R-module $\Bbbk$ (as defined in [7, Def. 3.2.3]) and define the 'Matlis dual' of an R-module M by setting

$$M^\vee := \mathrm{Hom}_R(M, E_R(\Bbbk))$$

(regarded as an R-module in the obvious way). We recall that $E_R(\Bbbk)$, and hence also $M^\vee$ for any given module M, is unique up to isomorphism. In addition, the injectivity $E_R(\Bbbk)$ implies that the assignment $M \mapsto M^\vee$ induces a contravariant exact functor on R-modules and, if M is either a finitely generated or Artinian R-module, then Matlis duality asserts that the natural map $M \to M^{\vee\vee}$ is bijective (cf. [7, Th. 3.2.13]).

Finally, we record a useful description of $E_R(\Bbbk)$ in terms of the filtration (2.6) (this result is essentially well-known but, for lack of a good reference, we include a proof).

(2.8) Lemma. *The following claims are valid.*

(i) For every n one can take $E_{R_n}(\Bbbk) = R_n$ and also fix an isomorphism of R_n-modules $R_n \cong \mathrm{Hom}_\Lambda(R_n, E_R(\Bbbk)) \cong E_R(\Bbbk)[\mathfrak{a}_n]$.

(ii) The isomorphisms in (i) induce an identification $T_n^\vee \cong T_n^*$ and also an injective homomorphism $R_n \cong E_R(\Bbbk)[\mathfrak{a}_n] \subseteq E_R(\Bbbk)[\mathfrak{a}_{n+1}] \cong R_{n+1}$ of R-modules. The module $E_R(\Bbbk)$ is isomorphic to the inductive limit $\varinjlim_{n\in\mathbb{N}} R_n$ with respect to the latter morphisms.

(iii) Let M be a finitely generated R_{n+1}-module. Then any choice of injection $R_n \hookrightarrow R_{n+1}$ as in (ii) induces an isomorphism $(M[\mathfrak{a}_n])^* \cong M^*/\mathfrak{a}_n M^*$.

Proof. At the outset we recall that, for any ring $\mathcal{R}$, an extension $N \subseteq M$ of $\mathcal{R}$-modules is called 'essential' if $U \cap N \neq (0)$ for every non-zero $\mathcal{R}$-submodule $U \subseteq M$. By [7, Prop. 3.2.2] one can then characterise injective $\mathcal{R}$-modules I as those $\mathcal{R}$-modules that have no essential extensions $I \subseteq M$ with $I \neq M$, and $E_{\mathcal{R}}(N)$ is then, by its definition, a maximal essential extension of N. The first assertion of (i) is proved in [7, Th. 3.2.10] and to prove the rest of (i) it is enough to construct an isomorphism of R-modules $E_{R_n}(\Bbbk) \cong E_R(\Bbbk)[\mathfrak{a}_n]$. To prove this, we note that $\Bbbk \subseteq E_\Lambda(\Bbbk)[\mathfrak{a}_n]$ is an essential extension of R_n-modules (since $\Bbbk \subseteq E_R(\Bbbk)$ is an essential extension of R-modules) and so $E_R(\Bbbk)[\mathfrak{a}_n]$ identifies with a submodule of $E_{R_n}(\Bbbk)$. In particular, $E_R(\Bbbk)[\mathfrak{a}_n]$ is finite and so it is enough to show $|E_R(\Bbbk)[\mathfrak{a}_n]| \geq |E_{R_n}(\Bbbk)|$. But this is clear since $\Bbbk \subseteq E_{R_n}(\Bbbk)$ is an essential extension of R-modules and so $E_{R_n}(\Bbbk)$ identifies with a submodule of $E_R(\Bbbk)[\mathfrak{a}_n]$. This proves (i).

The first isomorphism in (ii) is obtained via the following composite

$$\begin{aligned} T_n^\vee = \mathrm{Hom}_R(T_n, E_R(\Bbbk)) &= \mathrm{Hom}_R(T_n \otimes_R R_n, E_R(\Bbbk)) \\ &\cong \mathrm{Hom}_{R_n}(T_n, \mathrm{Hom}_R(R_n, E_R(\Bbbk))) \cong \mathrm{Hom}_{R_n}(T_n, R_n) = T_n^*. \end{aligned}$$

where the first isomorphism is induced by Tensor-Hom adjunction and the second by the isomorphism in (i). The second assertion of (ii) is clear and to prove the final assertion we note that $\varinjlim_{n\in\mathbb{N}} R_n$ is an essential extension of $\Bbbk$. Indeed, let $x \in \varinjlim_{n\in\mathbb{N}} R_n$, then $x \in R_n$ for some n and since R_n is an essential extension of $\Bbbk$, it follows that $(R_n x) \cap \Bbbk \neq 0$, as required. Now let

$E_R(\Bbbk)$ be a maximal essential extension of $\Bbbk$ that contains $\varinjlim_{n\in\mathbb{N}} R_n$. Then any element m of $E_R(\Bbbk)$ is annihilated by some ideal $\mathfrak{a}_n$: indeed, it is enough to show $\mathrm{Ann}_R(m)$ is $\mathfrak{m}$-primary, and this follows from the inclusion of sets of associated primes $\mathrm{Ass}_R(R/\mathrm{Ann}_R(m)) \subseteq \mathrm{Ass}_R(E_R(\Bbbk) = \{\mathfrak{m}\}$, where the last equality is by [7, Lem. 3.2.7 (a)]. Now, one has $E_R(\Bbbk)[\mathfrak{a}_n] = R_n$ by (i) and so we deduce that m belongs to $\varinjlim_{n\in\mathbb{N}} R_n$. It follows that $\varinjlim_{n\in\mathbb{N}} R_n$ is a maximal essential extension of $\Bbbk$, as required to prove (ii).

To prove (iii), we may fix an injective presentation $0 \to M \to R_{n+1}^{\oplus l} \xrightarrow{\cdot A} R_{n+1}^{\oplus m}$ for suitable integers $l, m \geq 0$ and an $(m\times l)$-matrix $A = (a_{ij})_{ij}$ because R_{n+1} is injective as an R_{n+1}-module. Since taking $\mathfrak{a}_n$-torsion defines a left-exact functor, dualising leads to an exact sequence

$$(R_{n+1}[\mathfrak{a}_n])^{\oplus m} \xrightarrow{\cdot A^{\mathrm{t}}} (R_{n+1}[\mathfrak{a}_n])^{\oplus l} \longrightarrow (M[\mathfrak{a}_n])^* \longrightarrow 0,$$

where $A^{\mathrm{t}} = (a_{ji})_{ij}$ is the transpose of A. Write $\sigma\colon R_n \hookrightarrow R_{n+1}$ for the fixed embedding, which restricts to an isomorphism $R_n \cong R_{n+1}[\mathfrak{a}_n]$ because R_n and $R_{n+1}[\mathfrak{a}_n] \cong \mathrm{Hom}_{R_{n+1}}(R_n, R_{n+1})$ have the same length as R_{n+1}-modules. The R_{n+1}-linearity of σ then implies that the diagram

$$\begin{array}{ccc} (R_{n+1}[\mathfrak{a}_n])^{\oplus m} & \xrightarrow{\cdot A^{\mathrm{t}}} & (R_{n+1}[\mathfrak{a}_n])^{\oplus l} \\ {\scriptstyle \sigma^{\oplus m}}\uparrow\simeq & & {\scriptstyle \sigma^{-1,\oplus l}}\downarrow\simeq \\ R_n^{\oplus m} & \xrightarrow{\quad\cdot A^{\mathrm{t}}\quad} & R_n^{\oplus l} \end{array}$$

is commutative. Since the cokernel of $R_n^{\oplus m} \xrightarrow{\cdot A^{\mathrm{t}}} R_n^{\oplus l}$ is canonically isomorphic to $M^*/\mathfrak{a}_n M^*$, we deduce that $\sigma^{-1,\oplus l}$ induces an isomorphism $(M[\mathfrak{a}_n])^* \xrightarrow{\simeq} M^*/\mathfrak{a}_n M^*$, as required. □

In connection with Lemma 2.8 the following general observation will be useful.

(2.9) Lemma. *Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$. For every non-zero element x in an R-module F there is an element $r \in R$ such that $r \cdot x$ is both non-zero and belongs to $F[\mathcal{M}] \coloneqq \{y \in F \mid a \cdot m = 0$ for all $a \in \mathcal{M}\}$.*

Proof. Since R is Artinian, the descending chain of R-modules $Rx \supseteq \mathcal{M}\cdot x \supseteq \cdots \supseteq \mathcal{M}^l \cdot x \supseteq \ldots$ becomes stationary and so there exists $i \in \mathbb{N}_0$ such that $\mathcal{M}^l \cdot x = \mathcal{M}^{l+1} \cdot x$. By Nakayama's lemma, this implies $\mathcal{M}^l \cdot x = 0$. Let l be the smallest non-negative integer with this property. Then, since $x \neq 0$ (by assumption), $l \geq 1$ and, by the minimality of l, $\mathcal{M}^{l-1} \cdot x$ is non-zero and contained in $F[\mathcal{M}]$. We may thus take r to be any element of $\mathcal{M}^{l-1} \setminus \mathcal{M}^l$. □

(2.10) Remark. Fix an element r as in Lemma 2.9 with $F = R$ and $x = 1$. If R is a zero-dimensional Gorenstein local ring, the ideal $R[\mathcal{M}]$ is a one-dimensional $\Bbbk$-vector space by Lemma 2.8 (i) and so, in particular, principal. The assignment $1 \mapsto r$ therefore induces an isomorphism of R-modules $\Bbbk = R/\mathcal{M} \cong R[\mathcal{M}]$.

See Lemma 2.32 below for a general result on base change over inverse limit rings.

2.2. Fitting ideals

In this section we fix a commutative unital ring $\mathcal{R}$ and a finitely-presented $\mathcal{R}$-module M. We then fix (as we may) a resolution of M

$$\mathcal{R}^{\oplus m} \xrightarrow{\varphi} \mathcal{R}^{\oplus n} \to M \to 0$$

in which $m \geq n$. For each $i \in \mathbb{N}_0$, the i-th 'Fitting ideal' $\mathrm{Fitt}^i_{\mathcal{R}}(M)$ of M is defined to be the ideal of $\mathcal{R}$ generated by the $(n-i)\times(n-i)$-minors of any matrix representing φ (where we use the convention that the determinant of an empty matrix is 1).

It is easily checked that the above definition is independent of the chosen resolution, that $\mathrm{Fitt}^i_{\mathcal{R}}(M) \subseteq \mathrm{Fitt}^j_{\mathcal{R}}(M)$ if $i \leq j$ and that $\mathrm{Fitt}^i_{\mathcal{R}}(M) = \mathcal{R}$ if $i \geq n$ (cf. [84, § 3.1, Th. 1, Th. 2]). In the following result we record several further properties of these ideals, most of which are well-known, that we shall rely on throughout this article. (A more detailed discussion, and further properties, of Fitting ideals can be found, for example, in each of [84, § 3.1], [78, App.] and [35, § 20.2].)

(2.11) Lemma. *The following claims are valid for all finitely-presented $\mathcal{R}$-modules M and N and all i and j in $\mathbb{N}_0$.*

(i) If $f\colon M \to N$ is a surjective map of $\mathcal{R}$-modules, then $\mathrm{Fitt}^i_{\mathcal{R}}(M) \subseteq \mathrm{Fitt}^i_{\mathcal{R}}(N)$.

(ii) If $0 \to N \to M \to M/N \to 0$ is a short exact sequence of $\mathcal{R}$-modules, then one has $\mathrm{Fitt}^i_{\mathcal{R}}(N) \cdot \mathrm{Fitt}^j_{\mathcal{R}}(M/N) \subseteq \mathrm{Fitt}^{i+j}_{\mathcal{R}}(M)$.

(iii) One has $\mathrm{Fitt}^i_{\mathcal{R}}(M \oplus N) = \sum_{a=0}^{a=i} \mathrm{Fitt}^a_{\mathcal{R}}(M) \cdot \mathrm{Fitt}^{i-a}_{\mathcal{R}}(N)$. In particular, if N is a non-zero free $\mathcal{R}$-module of rank r, then $\mathrm{Fitt}^i_{\mathcal{R}}(M) = \mathrm{Fitt}^{i+r}_{\mathcal{R}}(M \oplus N)$.

(iv) If $f\colon \mathcal{R} \to \mathcal{R}'$ is a morphism of rings, then $M \otimes_{\mathcal{R}} \mathcal{R}'$ is a finitely-presented $\mathcal{R}'$-module and $\mathrm{Fitt}^i_{\mathcal{R}'}(M \otimes_{\mathcal{R}} \mathcal{R}')$ is equal to the ideal of $\mathcal{R}'$ generated by $f(\mathrm{Fitt}^i_{\mathcal{R}}(M))$.

(v) (Buchsbaum–Eisenbud) If $j < i$, then $\mathrm{Fitt}^j_{\mathcal{R}}(M)$ annihilates $\bigwedge^i_{\mathcal{R}} M$. In particular, M is itself annihilated by $\mathrm{Fitt}^0_{\mathcal{R}}(M)$.

Proof. We fix a free presentation $\mathcal{R}^{\oplus m} \xrightarrow{\varphi} \mathcal{R}^{\oplus n} \to M \to 0$ and note that we have an exact sequence $0 \to \ker \pi \to \ker(f \circ \pi) \to \ker f \to 0$. We can therefore construct a free presentation $\mathcal{R}^{\oplus(m+m')} \xrightarrow{\varphi'} \mathcal{R}^{\oplus n} \xrightarrow{f\circ\pi} N \to 0$ in which φ' restricts to φ on $\mathcal{R}^{\oplus m}$. This implies (i) and the proof of (ii) is similar (see [84, § 3.1, Ex. 2, solution on p. 90] for the details). The first equality in (iii) is proved in [84, § 3.1, Ex. 3, solution on p. 92] and the second assertion follows as a consequence upon replacing i by $i + r$ in the first equality and noting that $\mathrm{Fitt}^j_{\mathcal{R}}(N)$ is equal to (0) if $j < r$ and to $\mathcal{R}$ if $j \geq r$. Claim (iv) follows directly from the definitions upon noting $(-) \otimes_{\mathcal{R}} \mathcal{R}'$ is a right-exact functor and hence that tensoring a free presentation of M with $\mathcal{R}'$ gives a free presentation of $M \otimes_{\mathcal{R}} \mathcal{R}'$.

The second assertion of (v) is classical and proved in [84, § 3.1, Th. 5]. It is also an immediate consequence of the first assertion with $i = 1$ and $j = 0$. To prove the first assertion of (v), we note that the general result of [8, Cor. 1.3] proves $\mathrm{Fitt}^j_{\mathcal{R}}(M)$ annihilates $\bigwedge^{j+1}_{\mathcal{R}} M$. (Note the different convention for Fitting ideals: in loc. cit. the authors write $F_k(M)$ for $\mathrm{Fitt}^{k-1}_{\mathcal{R}}(M)$ in our notation.) The claimed result then follows from the existence of a surjective map $(\bigwedge^{j+1}_{\mathcal{R}} M) \otimes_{\mathcal{R}} (\bigwedge^{i-j-1}_{\mathcal{R}} M) \twoheadrightarrow \bigwedge^i_{\mathcal{R}} M$. Since this result is perhaps not so well-known for the convenience of the reader we include its proof. Fix a free presentation $\mathcal{R}^{\oplus m} \xrightarrow{\varphi} \mathcal{R}^{\oplus n} \to M \to 0$ of M with $n \geq j + 1$. For every $s \geq 0$ we define the map

$$\varphi^{(s)}\colon \big(\bigwedge\nolimits^s_{\mathcal{R}} \mathcal{R}^{\oplus m}\big) \otimes_{\mathcal{R}} \bigwedge\nolimits^j_{\mathcal{R}} \mathcal{R}^{\oplus n} \to \bigwedge\nolimits^{j+s}_{\mathcal{R}} \mathcal{R}^{\oplus n}, \quad (x_1 \wedge \cdots \wedge x_s) \otimes y \mapsto \varphi(x_1) \wedge \cdots \wedge \varphi(x_s) \wedge y.$$

The starting point is then the exact seq

$$\mathcal{R}^{\oplus m} \otimes_{\mathcal{R}} \bigwedge\nolimits^j_{\mathcal{R}} \mathcal{R}^{\oplus n} \xrightarrow{\varphi^{(1)}} \bigwedge\nolimits^{j+1}_{\mathcal{R}} \mathcal{R}^{\oplus n} \longrightarrow \bigwedge\nolimits^{i+1}_{\mathcal{R}} M \longrightarrow 0,$$

which reduces us to proving that, for every $\lambda \in \mathrm{Fitt}^j_{\mathcal{R}}(M)$ and $b = b_1 \wedge \cdots \wedge b_{s+1} \in \bigwedge^{j+1}_{\mathcal{R}} \mathcal{R}^{\oplus n}$, one has that λb belongs to the image of the map $\varphi^{(1)}$.

Let $e_1, \ldots, e_{n-j-1} \in \mathcal{R}^{\oplus n}$ be basis vectors such that $\{b_1, \ldots, b_{j+1}, e_1, \ldots, e_{n-(j+1)}\}$ is a linearly independent set. Setting $e \coloneqq e_1 \wedge \cdots \wedge e_{n-j-1}$, one then has that $e^*(e \wedge b) = b$. Now, any choice of isomorphism $\bigwedge^n_{\mathcal{R}} \mathcal{R}^{\oplus n} \cong \mathcal{R}$ induces an identification

$$\mathrm{im}\Big(\big(\bigwedge\nolimits^{n-j}_{\mathcal{R}} \mathcal{R}^{\oplus m}\big) \otimes_{\mathcal{R}} \bigwedge\nolimits^j_{\mathcal{R}} \mathcal{R}^{\oplus n} \xrightarrow{\varphi^{(n-j)}} \bigwedge\nolimits^n_{\mathcal{R}} \mathcal{R}^{\oplus n} \cong \mathcal{R}\Big) \cong \mathrm{Fitt}^j_{\mathcal{R}}(M)$$

and so, in particular, $\mathrm{Fitt}^j_{\mathcal{R}}(M)$ annihilates the cokernel of $\varphi^{(n-j)}$. That is, $\lambda \cdot (e \wedge b)$ can be written in the form $\sum_{l\in[k]} \varphi(x_{l1}) \wedge \cdots \wedge \varphi(x_{l(n-j)}) \wedge y_l$ for suitable $x_{lp} \in \mathcal{R}^{\oplus m}$ and $y_i \in \bigwedge^j_{\mathcal{R}} \mathcal{R}^{\oplus n}$

with $p \in [k]$ and $l \in [n-j]$. It follows that

$$\begin{aligned}\lambda b &= e^*(\lambda \cdot (e \wedge b)) = e^*\big(\textstyle\sum_{l\in[k]}\varphi(x_{l1}) \wedge \cdots \wedge \varphi(x_{l(n-j)}) \wedge y\big) \\ &= \textstyle\sum_{l\in[k]}\sum_{p\in[n-j]} \pm\, \varphi(x_{lp}) \wedge e^*\big(\varphi(x_{l1} \wedge \cdots \wedge \widehat{\varphi(x_{lp})} \wedge \cdots \wedge \varphi(x_{lj}) \wedge y_l\big)\end{aligned}$$

belongs to the image of $\varphi^{(1)}$, as required. □

Finally, we establish a technical result about inverse limits of Fitting ideals that we need in later arguments. Claim (i) of this result extends the result [46, Th. 2.1] of Greither and Kurihara and (ii) extends an idea of Popescu and Yin from [88].

(2.12) Proposition. *Let $(\mathcal{R}_i, \rho_i)_{i\in\mathbb{N}}$ be a projective system of rings, and set $\mathcal{R} \coloneqq \varprojlim_{i\in\mathbb{N}}\mathcal{R}_i$. Let $(M_i, f_i)_{i\in\mathbb{N}}$ be a projective system of $\mathcal{R}_i$-modules with surjective transition maps f_i, and write M for the associated $\mathcal{R}$-module $\varprojlim_{i\in\mathbb{N}}M_i$. Assume that M is finitely presented and that, in addition, either of the following conditions are satisfied:*

(i) Each ring $\mathcal{R}_i$ (resp. module M_i) is a compact Hausdorff space and the transition maps ρ_i (resp. f_i) are continuous, and there exists a natural number s such that, for all $i \in \mathbb{N}$, the kernel of $M \otimes_{\mathcal{R}} \mathcal{R}_i \to M_i$ is generated by s elements over $\mathcal{R}_i$.

(ii) $\mathcal{R}$ satisfies condition (2.5), and $\mathcal{R}_i \coloneqq \mathcal{R}/\mathfrak{a}_i$ as in § 2.1.2.

Then, for every integer $r \geq 0$, one has $\mathrm{Fitt}^r_{\mathcal{R}}(M) = \varprojlim_{i\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}_i}(M_i)$, where the limit is taken with respect to the maps $\mathrm{Fitt}^r_{\mathcal{R}_{i+1}}(M_{i+1}) \to \mathrm{Fitt}^r_{\mathcal{R}_i}(M_i)$ induced by the restriction of ρ_i.

Proof. We first assume that the conditions in (i) are satisfied. Since each map $f_i\colon M_{i+1} \to M_i$ induces a surjection $M_{i+1} \otimes_{\mathcal{R}_{i+1}} \mathcal{R}_i \to M_i$, standard functorial properties of Fitting ideals implies that there is an inclusion

$$\rho_i(\mathrm{Fitt}^r_{\mathcal{R}_{i+1}}(M_{i+1})) = \mathrm{Fitt}^r_{\mathcal{R}_i}(M_{i+1} \otimes_{\mathcal{R}_{i+1}} \mathcal{R}_i) \subseteq \mathrm{Fitt}^r_{\mathcal{R}_i}(M_i).$$

In a similar way, since the second assertion of (i) implies that the natural map $M \otimes_{\mathcal{R}} \mathcal{R}_i \to M_i$ is surjective, one has $\theta_i(\mathrm{Fitt}^r_{\mathcal{R}}(M)) = \mathrm{Fitt}^r_{\mathcal{R}_i}(M \otimes_{\mathcal{R}} \mathcal{R}_i) \subseteq \mathrm{Fitt}^r_{\mathcal{R}_i}(M_i)$, with θ_i the natural projection map $\mathcal{R} \to \mathcal{R}_i$. By taking the limit over i, we therefore obtain an inclusion $\mathrm{Fitt}^r_{\mathcal{R}}(M) \subseteq \varprojlim_{i\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}_i}(M_i)$, and so we must prove the reverse inclusion.
To do this, we use the fact M is finitely presented to fix an exact sequence of $\mathcal{R}$-modules

$$\mathcal{R}^{t_1} \longrightarrow \mathcal{R}^{t_2} \longrightarrow M \longrightarrow 0$$

for suitable natural numbers t_1 and t_2. Now, each of the composite maps $\mathcal{R}^{t_2} \to M \to M_i$, where the second arrow is the natural projection, factors through $\mathcal{R}^{t_2} \to \mathcal{R}_i^{t_2}$. The assumption that the transition maps $f_j\colon M_{j+1} \to M_j$ are surjective ($j \geq i$) implies that $M \to M_i$ is surjective, hence we obtain a surjective map $\mathcal{R}_i^{t_2} \to M_i$. Setting $K_i \coloneqq \ker(\mathcal{R}_i^{t_2} \to M_i)$, we obtain an exact commutative diagram

$$\begin{array}{ccccc} \mathcal{R}_i^{t_1} & \longrightarrow & \mathcal{R}_i^{t_2} & \twoheadrightarrow & M \otimes_{\mathcal{R}} \mathcal{R}_i \\ \downarrow & & \| & & \downarrow \\ K_i & \hookrightarrow & \mathcal{R}_i^{t_2} & \twoheadrightarrow & M_i. \end{array} \tag{2.13}$$

The Snake Lemma, applied to the above diagram, then combines with condition (iii) to imply that K_i can be generated by at most $t_1 + s$ elements.
Since M_i is Hausdorff, K_i is a closed subspace of a compact Hausdorff space, hence is itself compact Hausdorff. The associated first derived limit therefore vanishes and so, passing to the limit of the bottom sequences in (2.13), we obtain the exact sequence

$$0 \longrightarrow K \coloneqq \varprojlim_{i\in\mathbb{N}}K_i \longrightarrow \mathcal{R}^{t_2} \longrightarrow M \longrightarrow 0.$$

Using this sequence, $\mathrm{Fitt}^r_{\mathcal{R}}(M)$ can then be computed as follows: For every subset $J \subseteq [t_2]$ of cardinality $t_2 - r$, we now write $\pi_J\colon \mathcal{R}^{t_2} \to \mathcal{R}^r$ for the projection $(x_1, \ldots, x_{t_2}) \mapsto (x_j)_{j\in J}$.

For each $v = (v_1, \ldots, v_r) \in K^r$, we then write $\pi_J(v)$ for the $(r \times r)$-matrix with columns $\pi_J(v_1), \ldots, \pi_J(v_r)$. Then $\mathrm{Fitt}^r_{\mathcal{R}}(M)$ is the ideal of $\mathcal{R}$ generated by all $\det(\pi_J(v))$ with v ranging over K^r and J ranging over all subsets of $[t_2]$ of cardinality r.

We now give a similar description of $\mathrm{Fitt}_{\mathcal{R}_i}(M_i)$ for every $i \in \mathbb{N}$. We write $\pi_{J,i} \colon \mathcal{R}_i^{t_2} \to \mathcal{R}_i^r$ for the projection onto coordinates in J, and, for every $v = (v_1, \ldots, v_r) \in K_i^r$, write $\pi_{J,i}(v)$ for the matrix with columns $\pi_{J,i}(v_1), \ldots, \pi_{J,i}(v_r)$. By definition, $\mathrm{Fitt}^r_{\mathcal{R}_i}(M_i)$ is then the ideal of $\mathcal{R}_i$ generated by all determinants of such matrices $\pi_{J,i}(v)$.

Since K_i can be generated by $t_2 + s$ elements and the determinant is multilinear, any matrix of the form above can be written as a sum of $N := (t_2 + s)^{t_2 - r}$ determinants with columns from $\pi_J(K_i)$. If we set $\mathcal{W}_i := \bigoplus_J (K_i^r)^N$, then we therefore have that $\mathrm{Fitt}^r_{\mathcal{R}_i}(M_i)$ is equal to the image of the map

$$\Phi_i \colon \mathcal{W}_i = \bigoplus\nolimits_J (K_i^r)^N \to \mathcal{R}_i, \quad (v_{J,j})_{J,j} \mapsto \sum\nolimits_J \sum\nolimits_{j \in [N]} \det(\pi_J(v_{J,j})),$$

where J runs over all subsets of $\{1, \ldots, t_2\}$ of cardinality $t_2 - r$.

We now claim that $\varprojlim_{i \in \mathbb{N}} \operatorname{im} \Phi_i$ is equal to the image of the map $\Phi \colon \varprojlim_{i \in \mathbb{N}} \mathcal{W}_i \to \mathcal{R}$ induced by $(\Phi_i)_{i \in \mathbb{N}}$. To show this, we suppose to be given an element $x = (x_i)_{i \in \mathbb{N}}$ of $\varprojlim_{i \in \mathbb{N}} \operatorname{im} \Phi_i$. For every $i \in \mathbb{N}$, we write $U_i := \Phi_i^{-1}(x_i)$ for the full preimage of x_i under Φ_i and note that each of the transition maps $\tilde{\rho}_{i,j} \colon \mathcal{W}_j \to \mathcal{W}_i$ takes U_j to U_i for $j \geq i$. Since $\mathcal{W}_j$ is finitely generated over the finite ring $\mathcal{R}_j$, it is again compact Hausdorff. As the preimage of a closed set under a continuous map, each U_j is then a closed subset of a compact Hausdorff, hence compact Hausdorff. Now, $\tilde{\rho}_{i,j}$ is continuous and so $\tilde{\rho}_{i,j}(U_j)$ is a compact subset of $\mathcal{W}_i$, which implies that it must be closed. In addition, the ascending chain given by the $\tilde{\rho}_{i,j}(U_j)$ has the finite intersection property because each preimage U_j is non-empty. It follows that the intersection $V_i := \bigcap_{j \geq i} \tilde{\rho}_{i,j}(U_j)$ is non-empty because $\mathcal{W}_i$ is compact. We now inductively construct a preimage $(w_i)_{i \in \mathbb{N}} \in \varprojlim_{i \in \mathbb{N}} \mathcal{W}_i$ of x with each w_i in V_i. For the induction base, we take w_1 to be any element of V_1. Now fix $i \geq 1$ and suppose that w_i is already constructed. We then define $U'_j := U_j \cap \tilde{\rho}_{i,j}^{-1}(w_i)$ for all $j > 1$ and note that each U'_j is non-empty because w_i belongs to V_i. Similar to before, it follows that the descending chain $(\tilde{\rho}_{i+1,j}(U'_j))_{j \geq i+1}$ has the finite intersection property, and hence that $V'_{i+1} := \bigcap_{j \geq i+1} \tilde{\rho}_{i+1,j}(U'_j)$ is non-empty. Any element w_{i+1} of V'_{i+1} will then both belong to V_i and have the property that $\tilde{\rho}_{i+1,i}(w_{i+1}) = w_i$, as required. This concludes the induction step.

We have thereby proved that any element x of $\varprojlim_{i \in \mathbb{N}} \operatorname{im} \Phi_i = \varprojlim_{i \in \mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}_i}(M_i)$ can be lifted to en element of $\varprojlim_{i \in \mathbb{N}} \mathcal{W}_i = \varprojlim_{i \in \mathbb{N}} \bigoplus_J (K_i^r)^N = \bigoplus_J (K^r)^N$. It follows that x is a finite sum of determinants of matrices of the form $\pi_J(v)$ with $v \in K^r$ and hence, by the explicit description given above, belongs to $\mathrm{Fitt}^r_{\mathcal{R}}(M)$. This proves the claimed equality under condition (i).

In the remainder of the argument, we therefore assume the validity of condition (ii) and we argue by induction on $d = \dim \mathcal{R}$. Since each projection map $M \to M_n$ is surjective, each $\mathcal{R}_n$-module M_n is finite and hence compact Hausdorff. The above argument therefore combines with Remark 2.16 below to prove the claim if $d \leq 1$. In the following we thus assume $d > 1$.

Following Remark 2.7, we obtain a filtration (2.6) of $\mathcal{R}$ by setting $\mathfrak{b}_n = (x_1^n, \ldots, x_d^n)$, and we now first justify that we may assume $\mathfrak{a}_n = \mathfrak{b}_n$. To do this we recall, as noted in Remark 2.7, one can define $f \colon \mathbb{N} \to \mathbb{N}$ such that $\mathfrak{b}_{f(n)} \subseteq \mathfrak{a}_n$ for all $n \in \mathbb{N}$. In particular, each M_n is a $\mathcal{R}/\mathfrak{b}_{f(n)}$-module and it suffices to prove that $\varprojlim_{n \in \mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{b}_{f(n)}}(M_n) = \varprojlim_{n \in \mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}_n}(M_n)$. Since the surjection $\mathcal{R}/\mathfrak{b}_{f(n)} \twoheadrightarrow \mathcal{R}_n = \mathcal{R}/\mathfrak{a}_n$ maps $\mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{b}_{f(n)}}(M_n)$ onto $\mathrm{Fitt}^r_{\mathcal{R}_n}(M_n)$, we have an exact sequence

$$0 \longrightarrow \mathfrak{a}_n/\mathfrak{b}_{f(n)} \longrightarrow \mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{b}_{f(n)}}(M_n) + (\mathfrak{a}_n/\mathfrak{b}_{f(n)}) \longrightarrow \mathrm{Fitt}^r_{\mathcal{R}_n}(M_n) \longrightarrow 0, \qquad (2.14)$$

in which all involved modules are finite. Now, taking the limit of the surjections $\mathfrak{a}_n \twoheadrightarrow \mathfrak{a}_n/\mathfrak{b}_{f(n)}$ gives a surjection $\varprojlim_{n \in \mathbb{N}} \mathfrak{a}_n \twoheadrightarrow \varprojlim_{n \in \mathbb{N}} (\mathfrak{a}_n/\mathfrak{b}_{f(n)})$, and this shows the vanishing of $\varprojlim_{n \in \mathbb{N}} (\mathfrak{a}_n/\mathfrak{b}_{f(n)})$ because $\varprojlim_{n \in \mathbb{N}} \mathfrak{a}_n = \bigcap_{n \in \mathbb{N}} \mathfrak{a}_n$ vanishes by the assumption that the natural map $\mathcal{R} \to \varprojlim_{n \in \mathbb{N}} \mathcal{R}/\mathfrak{a}_n$ is bijective. In a similar fashion, one shows that the natural map $\varprojlim_{n\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{b}_{f(n)}}(M_n) \to \varprojlim_{n\in\mathbb{N}}(\mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{b}_{f(n)}}(M_n) + (\mathfrak{a}_n/\mathfrak{b}_{f(n)}))$ is an isomorphism. Passing to the limit over the exact sequences (2.14) now shows that the natural map $\varprojlim_{n\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{b}_{f(n)}}(M_n) \to \varprojlim_{n\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}_n}(M_n)$ is bijective, as claimed.

In the remainder of this argument we therefore assume that $\mathfrak{a}_n = \mathfrak{b}_n = (x_1^n, \dots, x_d^n)$, and we also set $\mathfrak{a}'_n \coloneqq (x_1^n, \dots, x_{d-1}^n)$. Now, one has

$$\mathrm{Fitt}^r_{\mathcal{R}}(M) = \varprojlim_{n\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/(x_d^n)}(M\otimes_\Lambda(\Lambda/(x_d^n))) = \varprojlim_{n\in\mathbb{N}}\varprojlim_{m\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/(\mathfrak{a}'_m, x_d^n)}(M_m\otimes_{\mathcal{R}}(\mathcal{R}/(x_d^n))).$$

Here the first equality follows from the above argument (under condition (i)) and the second equality holds by our induction hypothesis. Since $M_m \cong \varprojlim_{n\in\mathbb{N}}(M_m \otimes_{\mathcal{R}} (\mathcal{R}/(x_d^n)))$ for all $m \in \mathbb{N}$, we may use the induction hypothesis again to deduce that

$$\varprojlim_{n\in\mathbb{N}}\varprojlim_{m\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/(\mathfrak{a}'_m, x_d^n)}(M_m \otimes_{\mathcal{R}} (\mathcal{R}/(x_d^n))) = \varprojlim_{m\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{a}'_m}(M_m).$$

Now, the image of $\mathrm{Fitt}^r_{\Lambda/\mathfrak{a}'_m}(M_m)$ in $\mathcal{R}_m$ is equal to $\mathrm{Fitt}^r_{\mathcal{R}_m}(M_m)$ by Lemma 2.11 (iv), so we have an exact sequence

$$0 \longrightarrow \mathfrak{a}_m/\mathfrak{a}'_m \longrightarrow \mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{a}'_m}(M_m) + (\mathfrak{a}_m/\mathfrak{a}'_m) \longrightarrow \mathrm{Fitt}^r_{\mathcal{R}_m}(M_m) \longrightarrow 0. \qquad (2.15)$$

Moreover, $\mathfrak{a}_m/\mathfrak{a}'_m$ is generated by x_d^m and so there exists a surjective map $x_d^m\Lambda \twoheadrightarrow \mathfrak{a}_m/\mathfrak{a}'_m$. Taking the limit over m then gives, because the involved modules are finitely generated $\mathcal{R}$-modules and hence compact Hausdorff, a surjection $\varprojlim_{m\in\mathbb{N}}(x_d^m\mathcal{R}) \twoheadrightarrow \varprojlim_{m\in\mathbb{N}}(\mathfrak{a}_m/\mathfrak{a}'_m)$. In addition, the Krull intersection theorem implies that the limit $\varprojlim_{m\in\mathbb{N}}(x_d^m\mathcal{R}) = \bigcap_{m\in\mathbb{N}}(x_d^m\mathcal{R})$ vanishes, and so the same is also true for $\varprojlim_{m\in\mathbb{N}}(\mathfrak{a}_m/\mathfrak{a}'_m)$. A similar argument shows that $\varprojlim_{m\in\mathbb{N}}\big((\mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{a}'_m}(M_m) + (\mathfrak{a}_m/\mathfrak{a}'_m))/\mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{a}'_m}(M_m)\big)$ vanishes. Taking the limit (over m) of the exact sequence (2.15) (in which all involved modules are compact Hausdorff) shows that

$$\varprojlim_{m\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}_m}(M_m) = \varprojlim_{m\in\mathbb{N}}\big(\mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{a}'_m}(M_m) + (\mathfrak{a}_m/\mathfrak{a}'_m)\big) = \varprojlim_{m\in\mathbb{N}} \mathrm{Fitt}^r_{\mathcal{R}/\mathfrak{a}'_m}(M_m),$$

as required to complete the proof. □

(2.16) Remark. The second condition in Proposition 2.12 (i) is automatically valid if $\mathcal{R}$ is local Noetherian and $\mathrm{Ann}_{\mathcal{R}}(M)$ contains an $\mathcal{R}$-regular sequence of length $\dim(\mathcal{R}) - 1$. (In particular, the condition is satisfied if $\dim \mathcal{R} \le 1$ or if $\dim(\mathcal{R}) = 2$ and M is $\mathcal{R}$-torsion.) To justify this, we note that the diagram (2.13) implies it is enough to prove the minimal number of $\mathcal{R}_i$-generators of K_i is bounded independently of i. If $\{x_i\}_{i\in[d-1]}$ is an $\mathcal{R}$-regular sequence in $\mathrm{Ann}_{\mathcal{R}}(M)$, then $(x_1, \dots, x_{d-1}) \cdot \mathcal{R}_i^{t_2} \subseteq K_i$. It therefore suffices to bound the minimal number of $\mathcal{R}_i$-generators of $K_i/(x_1, \dots, x_{d-1})\mathcal{R}_i^{t_2}$. The latter is now a submodule of $\mathcal{R}_i^{t_2}/(x_1, \dots, x_{d-1})\mathcal{R}_i^{t_2}$, and hence a subquotient of $(\mathcal{R}/(x_1, \dots, x_{d-1}))^{t_2}$. Since $\{x_i\}_{i\in[d-1]}$ is an $\mathcal{R}$-regular sequence, $\mathcal{R}/(x_1, \dots, x_{d-1})$ is a local Noetherian ring of Krull dimension one and so the claim follows from the fact that the minimal number of generators of any ideal of such a ring can be bounded by a constant that only depends on the ring (see, for example, [102]).

2.3. Exterior biduals

For any $r \in \mathbb{N}_0$, we write $\bigwedge_R^r M$ for the r-fold exterior power of an R-module M. The r-th 'exterior bidual' of M is then defined by setting

$$\bigcap\nolimits_R^r M = \Big(\bigwedge\nolimits_R^r M^*\Big)^*.$$

This construction is motivated by the approach of Rubin in [91, § 1.2] and is extensively studied in the literature (cf. [27, App. A], [95, App. B]). In this section, we establish several properties of these modules that will be useful in subsequent arguments.

2.3.1. Rank reduction

If $s \in \mathbb{N}_0$ is such that $s \le r$, and f is an element of $\bigwedge_R^s M^*$, then the assignment $a \mapsto [g \mapsto a(f \wedge g)]$ induces a 'rank reduction' map $\bigcap_R^r M \to \bigcap_R^{r-s} M$ which, by abuse of notation, we continue to denote by f.

(2.17) Lemma. *Let R be a G_2-ring and suppose to be given, for some $s \in \mathbb{N}$, an exact sequence*

$$0 \longrightarrow N \longrightarrow M \xrightarrow{(f_i)_{i\in[s]}} R^s \longrightarrow Z \longrightarrow 0$$

of finitely generated R-modules. Then the following claims are valid.

(i) For any natural number r, there exists an exact sequence

$$0 \longrightarrow \bigcap\nolimits_R^r N \longrightarrow \bigcap\nolimits_R^r M \xrightarrow{(f_i)_{i\in[s]}} \bigoplus\nolimits_{i\in[s]} \bigcap\nolimits_R^{r-1} M. \tag{2.18}$$

Here the first map is induced by the inclusion $N \to M$, and the second is the diagonal map induced by the maps $f_i\colon \bigcap_R^r M \to \bigcap_R^{r-1} M$ for each $i \in [s]$.

(ii) If $s \le r$, then the map $\wedge_{i\in[s]} f_i\colon \bigcap_R^r M \to \bigcap_R^{r-s} M$ factors through the map $\bigcap_R^{r-s} N \hookrightarrow \bigcap_R^{r-s} M$ in (2.18). In particular, there exists an induced map

$$\wedge_{i\in[s]} f_i\colon \bigcap\nolimits_R^r M \to \bigcap\nolimits_R^{r-s} N. \tag{2.19}$$

Proof. Claim (i) requires a slight variation of the proof of [95, Lem. B.12] and so, for clarification, we shall provide full details. Thus, by dualising the tautological exact sequence $0 \to N \to M \to M/N \to 0$, and setting $Y \coloneqq \mathrm{im}\{M^* \to N^*\}$, one obtains an exact sequence

$$0 \longrightarrow (M/N)^* \longrightarrow M^* \longrightarrow Y \longrightarrow 0. \tag{2.20}$$

This in turn implies the existence of an exact sequence

$$(M/N)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^* \longrightarrow \bigwedge\nolimits_R^r M^* \longrightarrow \bigwedge\nolimits_R^r Y \longrightarrow 0$$

(see [26, Lem. 2.5] for details) and, upon dualising again, this gives an exact sequence

$$0 \longrightarrow \big(\bigwedge\nolimits_R^r Y\big)^* \longrightarrow \bigcap\nolimits_R^r M \longrightarrow \big((M/N)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^*\big)^*. \tag{2.21}$$

We now first claim that $\big(\bigwedge_R^r Y\big)^* \cong \bigcap_R^r N$. To this end, we note that the cokernel of the inclusion $Y \subseteq N^*$ identifies with a submodule of $\mathrm{Ext}^1_R(M/N, R) \cong \mathrm{Ext}^2_R(Z, R)$, hence is pseudo-null because for every $\mathfrak{p} \in \mathrm{Spec}^1(R)$ the localisation $R_\mathfrak{p}$ has injective dimension at most one by condition (G_1) on R. It follows that both kernel and cokernel of the natural map $\bigwedge_R^r Y \to \bigwedge_R^r N^*$ are pseudo-null, which, by Lemma 2.4 (i), implies that the dual map $\bigcap_R^r N \to \big(\bigwedge_R^r Y\big)^*$ is an isomorphism, as claimed.

We now set $C \coloneqq \mathrm{coker}\{(R^s)^* \to (M/N)^*\}$ and observe that applying the right-exact functor $(-) \otimes_R \bigwedge_R^{r-1} M^*$ to the exact sequence $0 \to (R^s)^* \to X^* \to C \to 0$ gives an exact sequence

$$(R^s)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^* \longrightarrow (M/N)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^* \longrightarrow C \otimes_R \bigwedge\nolimits_R^{r-1} M^* \longrightarrow 0. \tag{2.22}$$

To investigate the last term in this sequence, we note that C naturally identifies with a submodule of $\mathrm{Ext}^1_R(Z, R)$. It follows from condition (G_0) on R that the latter , and hence also C, is a torsion R-module. By dualising (2.22) we therefore obtain an injection

$$\big((M/N)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^*\big)^* \hookrightarrow \big((R^s)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^*\big)^*.$$

Now, $(R^s)^*$ is a free R-module and so tensoring with $(R^s)^*$ commutes with taking duals. This combines with the isomorphism $(R^s)^{**} \cong R$ to give an isomorphism

$$\big((R^s)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^*\big)^* \cong R^s \otimes_R \bigcap\nolimits_R^{r-1} M.$$

By composing the last two displayed maps, we may then deduce the exact sequence (2.18) claimed in (i) from the exact sequence (2.21) upon recalling the isomorphism $\big(\bigwedge_R^r Y\big)^* \cong \bigcap_R^r N$. By the exactness of (2.18) it suffices to prove, for each $i \in [s]$, that the composite homomorphism $f_i \circ (\bigwedge_{j\in[s]} f_j)$ vanishes in order to verify (ii). For this, it is enough to note that, for each $a \in \bigcap_R^r M$, the element $(f_i \circ (\wedge_{j\in[s]} f_j))(a) = \big((\wedge_{j\in[s]} f_j) \wedge f_i\big)(a)$ vanishes since $f_i \wedge f_i = 0$. □

(2.23) Remark. R is a G_1-ring if and only if, for each finitely generated R-module M, the kernel of the canonical map $M \to M^{**}$ is the R-torsion submodule of M. Indeed, (S_1) is equivalent to asserting each prime in $\mathrm{Spec}^{\geq 1}(R)$ contains a nonzero divisor, and hence that R has no embedded primes, and so the claimed equivalence follows from [106, Th. A.1].

The next result extends [26, Prop. 2.4]. Before stating it, we recall that an R-module M is called 'torsion-less' if the natural map $M \to M^{**}$ is injective (cf. Bass [1, § 3]).

(2.24) Corollary. *Let R be a G_2-ring and $r \geq 1$ an integer. If N is an R-submodule of a finitely generated R-module M with the property that M/N is torsion-less, then one has*

$$\bigcap\nolimits_R^r N = \{a \in \bigcap\nolimits_R^r M \mid f(a) \in N^{**} \text{ for all } f \in \bigwedge\nolimits_R^{r-1} M^*\}.$$

Proof. Choose a surjection $R^{\oplus s} \twoheadrightarrow (M/N)^*$. Since M/N is assumed to be torsion-less, taking duals gives an injection $M/N \hookrightarrow (M/N)^{**} \hookrightarrow R^{\oplus s}$. We may therefore apply the argument of Lemma 2.17 (a) to the exact sequence $0 \to N \to M \to R^{\oplus s}$ to deduce the exact sequence (see the exact sequence (2.21))

$$0 \longrightarrow \bigcap\nolimits_R^r N \longrightarrow \bigcap\nolimits_R^r M \longrightarrow \big((M/N)^* \otimes_R \bigwedge\nolimits_R^{r-1} M^*\big)^*. \tag{2.25}$$

In order to show that an element a of $\bigcap_R^r M$ belongs to $\bigcap_R^r N$ it is therefore sufficient to prove that $a(m \wedge f) = 0$ for all $m \in (M/N)^*$ and $f \in \bigwedge_R^{r-1} M^*$. By definition, $f(a)$ is the map $x \mapsto a(x \wedge f)$, so $a(m \wedge f) = f(a)(m)$. Now suppose that $f(a)$ belongs to N^{**}, then the exact sequence (2.25) for $r = 1$ shows that $f(a)$ belongs to the kernel of the natural map $M^{**} \to (M/N)^{**}$ and so $f(a)(m)$ must vanish, as required. □

(2.26) Lemma. *Let R be a self-injective ring, and M a finitely generated R-module. For any integer $r \geq 0$ and $a \in \bigcap_R^r M$ one has $\mathrm{im}(a) = \mathrm{Ann}_R(\mathrm{Ann}_R(a))$.*

Proof. By definition, we have $\mathrm{im}(a) = \{a(f) \mid f \in \bigwedge_R^r M^*\}$. As R is self-injective, the map

$$\bigwedge\nolimits_R^r M^* \to \big(\bigcap\nolimits_R^r M\big)^*, \quad f \mapsto \{m \mapsto m(f)\}$$

is an isomorphism, and so $\mathrm{im}(a) = \{\varphi(a) \mid \varphi \in (\bigcap_R^r M)^*\}$. Now, R being self-injective also implies that any $\varphi \in (Ra)^*$ can be lifted to an element of $(\bigcap_R^r M)^*$, whence $\mathrm{im}(a) = \{\varphi(a) \mid \varphi \in (Ra)^*\}$. Since $Ra \cong R/\mathrm{Ann}_R(a)$, the claim now follows from the isomorphism

$$\mathrm{Hom}_R(R/\mathrm{Ann}_R(a), R) \xrightarrow{\simeq} \mathrm{Ann}_R(\mathrm{Ann}_R(a)), \quad f \mapsto f(1). \qquad \square$$

(2.27) Remark. A Noetherian ring R is self-injective if and only if, for every ideal I, one has $\mathrm{Ann}_R(\mathrm{Ann}_R(I)) = I$ (cf. [67, Th. 15.1]).

2.3.2. Reduced rings

In this subsection we assume to be given a reduced Noetherian ring $\mathcal{R}$. In this case the total ring of fractions $\mathcal{Q}$ of $\mathcal{R}$ is a finite product of fields and is therefore a semisimple, semilocal ring. This fact is used in the following result to interpret exterior biduals in terms of lattices considered by Rubin in [91, § 1.2].

(2.28) Lemma. *If M is a finitely generated $\mathcal{R}$-module, then, for any integer $r \geq 0$, the map ξ^r_M induces an isomorphism*

$$\big\{a \in \mathcal{Q} \otimes_{\mathcal{R}} {\bigwedge}^r_{\mathcal{R}} M \mid f(a) \in \mathcal{R} \text{ for all } f \in {\bigwedge}^r_{\mathcal{R}} M^* \big\} \xrightarrow{\simeq} {\bigcap}^r_{\mathcal{R}} M.$$

Proof. This argument closely follows that of [27, Prop. A.8] (which deals only with Gorenstein orders). Firstly, by applying the functor $\mathrm{Hom}_{\mathcal{R}}({\bigwedge}^r_{\mathcal{R}} M^*, -)$ to the tautological exact sequence $0 \to \mathcal{R} \to \mathcal{Q} \to \mathcal{Q}/\mathcal{R} \to 0$ one deduces that ${\bigcap}^r_{\mathcal{R}} M$ identifies with the kernel of the natural map $\mathrm{Hom}_{\mathcal{R}}\big({\bigwedge}^r_{\mathcal{R}} M^*, \mathcal{Q}\big) \to \mathrm{Hom}_{\mathcal{R}}\big({\bigwedge}^r_{\mathcal{R}} M^*, \mathcal{Q}/\mathcal{R}\big)$. In addition, since $\mathcal{Q}$ is semi-simple, the finitely generated $\mathcal{Q}$-module $\mathcal{Q} \otimes_{\mathcal{R}} M$ is projective and so there exists a natural composite isomorphism

$$\mathcal{Q} \otimes_{\mathcal{R}} {\bigwedge}^r_{\mathcal{R}} M \cong {\bigwedge}^r_{\mathcal{Q}} (\mathcal{Q} \otimes_{\mathcal{R}} M) \cong {\bigcap}^r_{\mathcal{Q}} (\mathcal{Q} \otimes_{\mathcal{R}} M) \cong \mathrm{Hom}_{\mathcal{R}}({\bigwedge}^r_{\mathcal{R}} M^*, \mathcal{Q}),$$

where the first isomorphism is clear, the second is $\xi^r_{\mathcal{Q} \otimes_{\mathcal{R}} M}$ and the third is induced by tensor-hom adjunction. The claimed equality is therefore true since an element a belongs to the kernel of the natural map $\mathcal{Q} \otimes_{\mathcal{R}} {\bigwedge}^r_{\mathcal{R}} M \to \mathrm{Hom}_{\mathcal{R}}\big({\bigwedge}^r_{\mathcal{R}} M^*, \mathcal{Q}/\mathcal{R}\big)$ if and only if one has $f(a) \in \mathcal{R}$ for every $f \in {\bigwedge}^r_{\mathcal{R}} M^*$. □

2.4. Complexes, resolutions, and determinants

In the sequel, for a (commutative) noetherian ring Λ, we write $D(\Lambda)$ for the derived category of Λ-modules. We also write $D^-(\Lambda)$ and $D^{\mathrm{perf}}(\Lambda)$ for the full triangulated subcategories of $D(\Lambda)$ comprising complexes that are respectively bounded above and 'perfect' (that is, isomorphic in $D(\Lambda)$ to a bounded complex of finitely generated projective Λ-modules).

Each object C of $D^{\mathrm{perf}}(\Lambda)$ has an associated Euler characteristic $\chi_\Lambda(C)$ in the Grothendieck group $K_0(\Lambda)$ of finitely generated projective Λ-modules. If Λ is local, then a finitely generated projective module is free and so the Λ-rank map $P \mapsto \mathrm{rk}_\Lambda(P)$ induces an isomorphism of $K_0(\Lambda)$ with $\mathbb{Z}$. In such cases, we regard $\chi_\Lambda(C)$ as an integer (so that the class of C in $K_0(\Lambda)$ is equal to $\chi_\Lambda(C) \cdot [\Lambda]$).

We write $D^{\mathrm{perf},0}(\Lambda)$ for the full triangulated subcategory of $D^{\mathrm{perf}}(\Lambda)$ represented by complexes C for which $\chi_\Lambda(C) = 0$. Let a and b be integers with $a \leq b$ and let $D^\bullet(\Lambda)$ denote either $D(\Lambda)$, $D^{\mathrm{perf}}(\Lambda)$ or $D^{\mathrm{perf},0}(\Lambda)$. Then we write $D^\bullet_{[a,b]}(\Lambda)$, $D^\bullet_{[a,\cdot]}(\Lambda)$ and $D^\bullet_{[\cdot,b]}(\Lambda)$ for the subcategories of $D^\bullet(\Lambda)$ that respectively comprise complexes C for which $H^i(C) = (0)$ if either $i < a$ or $i > b$, $H^i(C) = (0)$ if $i < a$ and $H^i(C) = (0)$ if $i > b$.

In the rest of this section, we fix a ring $R \cong \varprojlim_{n \in \mathbb{N}} R_n$ as in § 2.1.2 (so that, in particular, R is Noetherian and local) and describe two useful constructions of complexes in $D^{\mathrm{perf}}(R)$.

2.4.1. Limit complexes

The following result is essentially well-known.

(2.29) Lemma. *We fix integers a and b with $a < b$ and, for every natural number n, assume to be given data of the following form*

(a) an object C_n of $D^{\mathrm{perf}}_{[a,b]}(R_n)$;

(b) an isomorphism $\theta_n \colon R_n \otimes^{\mathbb{L}}_{R_{n+1}} C_{n+1} \cong C_n$ in $D(R_n)$.

Then there exists a bounded complex $\varprojlim_n C_n$ of finitely generated free R-modules that is unique up to isomorphism in $D^{\mathrm{perf}}(R)$ and has the following properties:

(i) $(\varprojlim_n C_n)^i = (0)$ unless $a \leq i \leq b$;

(ii) for every m, there exists an isomorphism $R_m \otimes^{\mathbb{L}}_R (\varprojlim_n C_n) \cong C_m$ in $D(R_m)$;

(iii) for every i, the induced composite map

$$H^i(\varprojlim_n C_n) \to \varprojlim_n H^i(R_n \otimes^{\mathbb{L}}_R (\varprojlim_m C_m)) \cong \varprojlim_n H^i(C_n)$$

is bijective.

Further, if $R_m\otimes^{\mathbb{L}}_{\Lambda}C$ belongs to $D^{\rm perf,0}(R_m)$ for any given m, then $\varprojlim_n C_n$ belongs to $D^{\rm perf,0}(R)$.

Proof. The conditions (a) and (b) implies that the general argument of [48, XV, p. 472 to the end] applies to realise each C_n by a bounded complex $\hat{C}_n^\bullet$ of finitely generated R_n-modules with $\hat{C}_n^i=(0)$ if either $i<a$ or $i>b$, $\hat{C}_n^i$ free for all $i\neq a$ and $\hat{C}_n^a$ of finite projective dimension, and each isomorphism θ_n by an isomorphism of complexes $\hat{\theta}_n\colon R_n\otimes_{R_{n+1}}\hat{C}^\bullet_{n+1}\cong\hat{C}^\bullet_n$ in such a way that the following condition is satsified. If $(\tilde{C}^\bullet_n,\tilde{\theta}_n)$ is any other family of complexes and isomorphisms that satisfy the same conditions, then there exists a family $(\phi_n\colon\hat{C}^\bullet_n\to\tilde{C}^\bullet_n)_n$ of isomorphisms of complexes of R_n-modules with the property that, for every n, there exists a commutative diagram of morphisms of complexes of R_n-modules

$$\begin{array}{ccc}\hat{C}^\bullet_{n+1} & \xrightarrow{\phi_{n+1}} & \tilde{C}^\bullet_{n+1}\\ \downarrow & & \downarrow\\ \hat{C}^\bullet_n & \xrightarrow{\phi_n} & \tilde{C}^\bullet_n\end{array} \tag{2.30}$$

in which the first and second vertical morphism are respectively induced by $\hat{\theta}_n$ and $\tilde{\theta}_n$.
We next claim that each R_n-module $\hat{C}^a_n$ is projective, and hence free. To see this, we note that for each $\mathfrak{p}\in{\rm Spec}(R_n)$, one has ${\rm depth}_{R_{n,\mathfrak{p}}}(R_{n,\mathfrak{p}})=\dim(R_{n,\mathfrak{p}})=0$ (as R_n is Cohen–Macaulay), so ${\rm pd}_{R_{n,\mathfrak{p}}}(\hat{C}^a_{n,\mathfrak{p}})=0$ (by the Auslander–Buchsbaum formula) and hence $\hat{C}^a_{n,\mathfrak{p}}$ is a free $R_{n,\mathfrak{p}}$-module. It follows that $\hat{C}^a_n$ is a locally free R_n-module and hence projective, as claimed.
The limit $C:=\varprojlim_n\hat{C}^\bullet_n$ with respect to the morphisms $\hat{C}^\bullet_{n+1}\to R_n\otimes_{R_{n+1}}\hat{C}^\bullet_{n+1}\cong\hat{C}^\bullet_n$ induced by $\hat{\theta}_n$ is then a bounded complex of R-modules that is unique up to isomorphism in $D(R)$ (as a consequence of the diagrams (2.30)) and for which there exists for every n a natural isomorphism $R_n\otimes_R C\cong\hat{C}^\bullet_n$ of complexes of R_n-modules.
We now claim that each term $C^i=\varprojlim_n\hat{C}^i_n$ is a finitely generated free R-module, and hence that C belongs to $D^{\rm perf}(R)$. To see this, we note each R_n-module $\hat{C}^i_n$ is finitely generated and free and that each projection map $\hat{C}^i_{n+1}\to R_n\otimes_{R_{n+1}}\hat{C}^i_{n+1}\cong\hat{C}^i_n$ is surjective and has kernel $(\mathfrak{a}_n/\mathfrak{a}_{n+1})\hat{C}^i_{n+1}$. Thus, since $\mathfrak{a}_n/\mathfrak{a}_{n+1}$ is contained in ${\rm Jac}(R_{n+1})$, Nakayama's Lemma implies that each limit $C^i=\varprojlim_n\hat{C}^i_n$ is finitely generated and free over $R=\varprojlim_n R_n$ with ${\rm rk}_R(C^i)={\rm rk}_{R_m}(\hat{C}_m)$ for any choice of m.
Next we note that each group $H^i(C_n)$ is finite and hence, since inverse systems of finite groups satisfy the Mittag–Leffler condition, the first derived limit $\varprojlim^{(1)}_n H^i(C_n)$ vanishes. From the exact sequences $0\to\varprojlim^{(1)}_n H^{i-1}(C_n)\to H^i(C)\to\varprojlim_n H^i(\hat{C}_n)\to 0$ we then deduce that the natural maps $H^i(C)\to\varprojlim_n H^i(\hat{C}_n)=\varprojlim_n H^i(C_n)$ are bijective, as claimed in (c).
It thus only remains to show that $\chi_R(C)=0$ if $\chi_{R_m}(R_m\otimes^{\mathbb{L}}_{\Lambda}C)=0$ for any given m. But this is true, since, by the argument above, each R-module C^i is free of rank ${\rm rk}_{R_m}(\hat{C}^i_m)$, whilst, as R_m is Cohen-Macauley, the Bass Cancellation Theorem [113, Th. I.2.3 (b)] combines with the vanishing of $\chi_{R_m}(R_m\otimes_R C)=\chi_{R_m}(\hat{C}^\bullet_m)$ to imply that $\sum_{i\in\mathbb{Z}}(-1)^i{\rm rk}_{R_m}(\hat{C}^i_m)=0$. □

We next record relations between the cohomology modules of complexes C_n as in Lemma 2.29.

(2.31) Lemma. *Fix integers a and b with $a<b$ and, for every natural number n, assume the pair (C_n,θ_n) satisfies (a) and (b) in Lemma 2.29. Then the following claims are valid.*

- *(i) Any choice of R_{n+1}-linear injection $j_n\colon R_n\hookrightarrow R_{n+1}$ of rings as in Lemma 2.8 induces an isomorphism $H^a(C_{n+1})[\mathfrak{a}_{n+1}]\cong H^a(C_n)$.*
- *(ii) Each map θ_{n+1} induces an isomorphism $H^b(C_{n+1})\otimes_{R_{n+1}}R_n\cong H^b(C_n)$.*

Proof. We have seen in the proof of Lemma 2.29 that C_{n+1} can be represented by a bounded complex $\hat{C}^\bullet_n$ of finitely generated free R_{n+1}-modules $\hat{C}^i_n$ that are zero if $i\notin[a,b]$ and with differentials $\partial^i_{n+1}\colon\hat{C}^i_{n+1}\to\hat{C}^{i+1}_{n+1}$. In particular, $H^a(C_{n+1})$ identifies with the kernel of ∂^a_{n+1}

and so we may construct a commutative diagram of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^a(C_{n+1})[\mathfrak{a}_{n+1}] & \longrightarrow & \hat{C}^a_{n+1}[\mathfrak{a}_{n+1}] & \xrightarrow{\partial^a_{n+1}} & \hat{C}^{a+1}_{n+1} \\ & & \uparrow & & \simeq\uparrow & & \simeq\uparrow \\ 0 & \longrightarrow & H^a(C_{n+1}\otimes^{\mathbb{L}}_{R_{n+1}} R_n) & \longrightarrow & C^a_{n+1}\otimes_{R_{n+1}} R_n & \xrightarrow{\partial^a_{n+1}} & C^{a+1}_{n+1}\otimes_{R_{n+1}} R_n, \end{array}$$

where the first row is exact because $(-)[\mathfrak{a}_{n+1}]$ is a left-exact functor, the two vertical isomorphisms are induced by our choice of identification $j_n\colon R_n \cong R_{n+1}[\mathfrak{a}_{n+1}]$, and the second square commutes because j_n is R_{n+1}-linear. Since θ_n identifies C_n with $\hat{C}^\bullet_{n+1}\otimes^{\mathbb{L}}_{R_{n+1}} R_n$, the dashed arrow in the above diagram gives the isomorphism required to establish claim (i).
As for claim (ii), we note that $H^b(C_{n+1})$ identifies with the cokernel of ∂^b_{n+1} because $\hat{C}^i_{n+1}$ vanishes for $i > b$. Now, $C_n \cong \hat{C}^\bullet_{n+1}\otimes^{\mathbb{L}}_{R_{n+1}} R_n$ via θ_n and so $H^1(C_n)$ can similarly be identified with the cokernel of the map $\hat{C}^{b-1}_{n+1}\otimes_{R_{n+1}} R_n \to \hat{C}^b_{n+1}\otimes_{R_{n+1}} R_n$ induced by ∂^b_{n+1}. Given this, (ii) follows upon noting these two cokernels agree since $(-)\otimes_{R_{n+1}} R_n$ is a right-exact functor. □

We finally prove a useful technical result concerning Tor-groups.

(2.32) Lemma. *Let $f\colon R \to S$ be a morphism of rings satisfying (2.5), and assume that f arises as the limit of commutative diagrams of finite rings of the form*

$$\begin{array}{ccc} R_{n+1} & \xrightarrow{f_{n+1}} & S_{n+1} \\ \downarrow & & \downarrow \\ R_n & \xrightarrow{f_n} & S_n. \end{array}$$

Let $(M_n)_{n\in\mathbb{N}}$ be a projective system of R_n-modules and set $M \coloneqq \varprojlim_{n\in\mathbb{N}} M_n$.

(i) For every $n \in \mathbb{N}$ such that M_{n+1} is finitely presented as an R_{n+1}-module, the maps f_n and $M_{n+1} \to M_n$ induce a natural map

$$\mathrm{Tor}_1^{R_{n+1}}(M_{n+1}, S_{n+1}) \to \mathrm{Tor}_1^{R_n}(M_n, S_n),$$

where we regard each S_n as an R_n-module via f_n. Further, if f_{n+1} and f_n are surjective and the natural map $M_{n+1}\otimes_{R_{n+1}} R_n \to M_n$ is bijective, then the cokernel of the above map is isomorphic to the cokernel of the natural map

$$\mathrm{Tor}_1^{R_{n+1}}(M_{n+1}, R_n) \to \mathrm{Tor}_1^{S_{n+1}}(M_{n+1}\otimes_{R_{n+1}} S_{n+1}, S_n). \qquad (2.33)$$

(ii) If all transition maps $M_{n+1} \to M_n$ are surjective, and M is finitely presented as an R-module, then there exists a canonical isomorphism

$$\varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_1^{R_n}(M_n, S_n) \cong \mathrm{Tor}_1^R(M, S),$$

where the limit is defined with respect to the maps in (i).

Proof. At the outset we note that if $\mathcal{R} \to \mathcal{S}$ is a morphism of commutative Noetherian rings and C is a perfect complex of $\mathcal{R}$-modules, then one has a spectral sequence

$$E_2^{i,j} = \mathrm{Tor}^{\mathcal{R}}_{-i}(H^j(C), \mathcal{S}) \Rightarrow E^{i+j} = H^{i+j}(C\otimes^{\mathbb{L}}_{\mathcal{R}} \mathcal{S}).$$

If C is acyclic outside degrees zero and one, this spectral sequence degenerates on its second page to give an exact sequence

$$\mathrm{Tor}_2^{\mathcal{R}}(H^1(C), \mathcal{S}) \longrightarrow H^0(C)\otimes_{\mathcal{R}} \mathcal{S} \longrightarrow H^0(C\otimes^{\mathbb{L}}_{\mathcal{R}} \mathcal{S}) \longrightarrow \mathrm{Tor}_1^{\mathcal{R}}(H^1(C), \mathcal{S}) \longrightarrow 0. \qquad (2.34)$$

Choose an R_{n+1}-free presentation $P^0_{n+1} \to P^1_{n+1} \to M_{n+1} \to 0$ of M_{n+1}, which we regard as a perfect complex $P_{n+1} = [P^0_{n+1} \to P^1_{n+1}]$ in $D(R_{n+1})$, where P^0_{n+1} is placed in degree zero. We then obtain a commutative diagram of the form

$$\begin{array}{ccccc} H^0(P_{n+1})\otimes_{R_{n+1}} S_{n+1} & \longrightarrow & H^0(P_{n+1}\otimes^{\mathbb{L}}_{R_{n+1}} S_{n+1}) & \twoheadrightarrow & \mathrm{Tor}_1^{R_{n+1}}(M_{n+1}, S_{n+1}) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(P_{n+1}\otimes^{\mathbb{L}}_{R_{n+1}} R_n)\otimes_{R_n} S_n & \longrightarrow & H^0(P_{n+1}\otimes^{\mathbb{L}}_{R_{n+1}} S_n) & \twoheadrightarrow & \mathrm{Tor}_1^{R_n}(M_{n+1}\otimes_{R_{n+1}} R_n, S_n). \end{array}$$

Here the first two vertical arrows are the natural maps, the first row is (2.34) applied with $C = P_{n+1}$, $\mathcal{R} = R_{n+1}$, and $\mathcal{S} = S_{n+1}$, and the second row is (2.34) applied with $C = P_{n+1} \otimes^{\mathbb{L}}_{R_{n+1}} R_n$, $\mathcal{R} = R_n$, and $\mathcal{S} = S_n$. Since the first square commutes, the dashed arrow now exists due to exactness of the rows, and we define the map in claim (i) to be the composite of this dashed arrow with the natural map $\mathrm{Tor}_1^{R_n}(M_{n+1} \otimes_{R_{n+1}} R_n, S_n) \to \mathrm{Tor}_1^{R_n}(M_n, S_n)$ induced by $M_{n+1} \otimes_{R_{n+1}} R_n \to M_n$.

To complete the proof of (i), we assume that the maps f_{n+1} and f_n are surjective and the map $M_{n+1} \otimes_{R_{n+1}} R_n \to M_n$ is bijective. In this case, the above diagram simplifies to give an exact commutative diagram

$$\begin{array}{ccccc}
H^0(P_{n+1}) & \longrightarrow & H^0(P_{n+1} \otimes^{\mathbb{L}}_{R_{n+1}} S_{n+1}) & \twoheadrightarrow & \mathrm{Tor}_1^{R_{n+1}}(M_{n+1}, S_{n+1}) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(P_{n+1} \otimes^{\mathbb{L}}_{R_{n+1}} R_n) & \longrightarrow & H^0(P_{n+1} \otimes^{\mathbb{L}}_{R_{n+1}} S_n) & \twoheadrightarrow & \mathrm{Tor}_1^{R_n}(M_n, S_n) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
\mathrm{Tor}_1^{R_{n+1}}(M_{n+1}, R_n) & \xrightarrow{\alpha} & \mathrm{Tor}_1^{S_{n+1}}(M_{n+1} \otimes_{R_{n+1}} S_{n+1}, S_n). & &
\end{array}$$

Here the exactness of the first two columns follows from an application of (2.34) with $C = P_{n+1}$, $\mathcal{R} = R_{n+1}$, and $\mathcal{S} = S_n$, respectively $C = P_{n+1} \otimes^{\mathbb{L}}_R S_{n+1}$, $\mathcal{R} = S_{n+1}$, and $\mathcal{S} = S_n$, and α is the natural map in (2.33). The final assertion of (i) is therefore obtained by applying the Snake Lemma to this diagram.

To prove (ii), we fix an R-free presentation $P^0 \to P^1 \to M \to 0$ and regard it as a complex $P = [P^0 \to P^1]$ in $D(R)$. Taking $P_{n+1} = P \otimes^{\mathbb{L}}_R R_{n+1}$ and $M_{n+1} = M \otimes_R R_{n+1}$ in the first diagram above and passing to the limit over n leads to an exact sequence (because all appearing modules are assumed to be finite) that forms the first row of the diagram

$$\begin{array}{ccccccc}
\varprojlim_{n\in\mathbb{N}} H^0(P \otimes^{\mathbb{L}}_R R_n) & \longrightarrow & \varprojlim_{n\in\mathbb{N}} H^0(P \otimes^{\mathbb{L}}_R S_n) & \twoheadrightarrow & \varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_1^{R_n}(M_n, S_n) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \vdots & & \\
H^0(P) & \longrightarrow & H^0(P \otimes^{\mathbb{L}}_R S) & \twoheadrightarrow & \mathrm{Tor}_1^R(M, S) & \longrightarrow & 0.
\end{array}$$

Here the second row is (2.34) applied with $C = P$, $\mathcal{R} = R$, and $\mathcal{S} = S$, and the isomorphisms are by Lemma 2.29. As a consequence, we deduce that the dashed arrow is an isomorphism.

For every $n \in \mathbb{N}$, we now write K_n for the kernel of the map $M \otimes_R R_n \to M_n$ (which is surjective because the system $(M_n)_{n\in\mathbb{N}}$ is assumed to have surjective transition maps). Since all appearing modules in the exact sequence

$$\mathrm{Tor}_1^{R_n}(K_n, S_n) \longrightarrow \mathrm{Tor}_1^{R_n}(M \otimes_R R_n, S_n) \longrightarrow \mathrm{Tor}_1^{R_n}(M_n, S_n) \longrightarrow K_n \otimes_{R_n} S_n$$

are finite, passing to the limit over n yields an exact sequence. To prove the isomorphism claimed in (ii), it is therefore enough to show $\varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_1^{R_n}(K_n, S_n)$ and $\varprojlim_{n\in\mathbb{N}}(K_n \otimes_{R_n} S_n)$ both vanish. To do this, we note that the limit over the maps $M \otimes_R R_n \to M_n$ is an isomorphism (because M is finitely presented and each R_n is finite), and hence that $\varprojlim_{n\in\mathbb{N}} K_n$ vanishes. We now fix $m \in \mathbb{N}$, write g_n for the map $K_n \to K_m$ for every $n \geq m$, and consider the group of universal norms $U_m \coloneqq \bigcap_{n\geq m} f_n(K_n)$ in K_m. Since each K_n is finite, U_m coincides with the image of $\varprojlim_{n\in\mathbb{N}} K_n \to K_m$ (cf. [12, Lem. 3.10]), which implies that $U_m = (0)$. By finiteness of K_m, there is then $N \geq m$ such that f_N is the zero map. It follows that also the maps $\mathrm{Tor}_1^{R_N}(K_N, S_N) \to \mathrm{Tor}_1^{R_m}(K_m, S_m)$ and $K_N \otimes_{R_N} S_N \to K_m \otimes_{R_m} S_m$ induced by g_N are both zero. As m was chosen arbitrary, this proves the required vanishing of $\varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_1^{R_n}(K_n, S_n)$ and $\varprojlim_{n\in\mathbb{N}}(K_n \otimes_{R_n} S_n)$, thereby concluding the proof of (ii). □

2.4.2. Quadratic resolutions

The class of complexes considered in the next result plays an important role in later arguments.

(2.35) Lemma. *Let C be an object of $D^{\mathrm{perf}}_{[\cdot,1]}(R)$ such that, for all n, $R_n \otimes^{\mathbb{L}}_R C$ belongs to $D^{\mathrm{perf}}_{[0,\cdot]}(R_n)$. Then the following claims are valid.*

(i) The integer $a \coloneqq \chi_{R_n}(R_n \otimes^{\mathbb{L}}_R C)$ is independent of n.

(ii) C is isomorphic in $D^{\mathrm{perf}}(R)$ to a complex of finitely generated free R-modules $P_0 \xrightarrow{\phi} P_1$, in which P_0 occurs in degree 0 and one has $\chi_R(C) = \mathrm{rk}_R(P_0) - \mathrm{rk}_R(P_1) = a$.

(iii) Let Y be a free R-module quotient of $H^1(C)$. Then the composite map $\pi\colon P_1 \to \mathrm{cok}(\phi) \cong H^1(C) \to Y$ induced by (ii) induces an isomorphism of R-modules $P_1 \cong \ker(\pi) \oplus Y$. In particular, the R-module $\ker(\pi)$ is free of rank $\mathrm{rk}_R(P_1) - \mathrm{rk}_R(Y)$.

Proof. Claim (i) is clear, and we then define a complex

$$\tilde{C} \coloneqq \begin{cases} C \oplus R^{\oplus a}[-1], & \text{if } a \geq 0, \\ C \oplus R^{\oplus(-a)}[0], & \text{if } a < 0. \end{cases}$$

This complex belongs to $D^{\mathrm{perf}}_{[\cdot,1]}(R)$ and, for all n, $R_n \otimes^{\mathbb{L}}_R \tilde{C}$ belongs to $D^{\mathrm{perf}}_{[0,\cdot]}(R_n)$. It is also clear that the validity of (ii) and (iii) for $\tilde{C}$ implies their validity for C. Hence, after replacing C by $\tilde{C}$ if necessary, in the rest of the argument we assume $a = 0$.

We first consider the case that R is finite and self-injective. Then, after fixing a surjective map of R-modules $\tau\colon P \to H^1(C)$, with P finitely generated and free, the argument of [27, Prop. A.11 (i)] proves the existence of a finitely generated R-module P' of finite projective dimension such that C is isomorphic to $P' \xrightarrow{\phi} P$ and the induced map $P \to \mathrm{cok}(\phi) \cong H^1(C)$ coincides with τ. Since the vanishing of $\chi_R(C)$ combines with the argument of Lemma 2.29 to imply P' is isomorphic to P, the claimed result is therefore true in this case.

To deal now with the general case, we fix a surjective map of R-modules $\tau\colon P \to H^1(C)$, with P finitely generated and free and note that, as Y is free, the composite τ' of τ with the projection $H^1(C) \to Y$ induces an isomorphism $P \cong \ker(\tau') \oplus Y$. For each n, we set $P_n \coloneqq R_n \otimes_R P$ and $C_n \coloneqq R_n \otimes^{\mathbb{L}}_R C$ and note $H^1(C_n) \cong R_n \otimes_R H^1(C)$ since $H^i(C) = (0)$ for all $i > 1$. With τ_n denoting the surjective map $P_n \to H^1(C_n)$ induced by τ, we can then combine the argument in the first paragraph with that of Lemma 2.29 to fix a resolution $P_n \xrightarrow{\phi_n} P_n$ of C_n so that the induced map $P_n \to \mathrm{cok}(\phi_n) \cong H^1(C_n)$ is τ_n and the isomorphism $R_n \otimes^{\mathbb{L}}_{R_{n+1}} C_{n+1} \cong C_n$ in $D(R_n)$ is induced by the identification $R_n \otimes_{R_{n+1}} P_{n+1} = R_n \otimes_R P = P_n$. Setting $\phi \coloneqq \varprojlim_n \phi_n$, one can then check that the complex $P \xrightarrow{\phi} P$ is isomorphic in $D(R)$ to C in such a way that the induced map $P \to \mathrm{cok}(\phi) \cong H^1(C)$ coincides with $\tau = \varprojlim_n \tau_n$. One then easily verifies that the complex $P \xrightarrow{\phi} P$ has all of the required properties. □

In the following two results, we record some useful consequences of the quadratic resolutions constructed in Lemma 2.35(i).

In both of these results, we fix data C and Y as in the last result, we write X for the kernel of the (given) surjective homomorphism $H^1(C) \to Y$ and we set

$$r = r_{Y,C} \coloneqq \mathrm{rk}_R(Y) + \chi_R(C).$$

(2.36) Lemma. *Fix $i \in \mathbb{N}_0$ such that $i + r \geq 0$. Then one has*

$$\mathrm{Fitt}^{i+r}_R(H^1(\mathrm{RHom}_R(C,R)[-1])) = \mathrm{Fitt}^i_R(X)$$

and hence, if R is self-injective, $\mathrm{Fitt}^{i+r}_R(H^0(C)^*) = \mathrm{Fitt}^i_R(X)$.

Proof. By using Lemma 2.11 (iii), it is easily checked that the stated claims are valid if and only if they are valid after replacing C by the complex $\tilde{C}$ used in the proof of Lemma 2.35.

In particular, since $\chi_R(\tilde{C}) = 0$, in the sequel we shall assume $\chi_R(C) = 0$, and hence that $r = \mathrm{rk}_R(Y)$.
Now, since Y is free, there exists a (non-canonical) isomorphism of R-modules $H^1(C) \cong X \oplus Y$ and so, by using Lemma 2.11 (iii) again, one finds that $\mathrm{Fitt}^i_R(X) = \mathrm{Fitt}^{i+r}_R(H^1(C))$. It therefore suffices to prove that $\mathrm{Fitt}^j_R(H^1(\mathrm{RHom}_R(C,R)[-1])) = \mathrm{Fitt}^j_R(H^1(C))$ for each $j \geq 0$.
To do this we note that, since $\chi_R(C) = 0$, Lemma 2.35 (ii) implies that we can fix a resolution of C of the form $P \xrightarrow{\phi} P$, with the first term placed in degree 0. We set $n \coloneqq \mathrm{rk}_R(P)$ and write A for the $(n \times n)$-matrix representing ϕ with respect to a fixed choice of basis $\mathfrak{B}$ for P. Then $\mathrm{Fitt}^j_R(H^1(C))$ can be computed as the ideal of R that is generated by the collection of $(n-j)\times(n-j)$-minors of A. We write $b^* \colon P \to R$ for the 'dual' of $b \in \mathfrak{B}$ and define a basis of P^* by $\mathfrak{B}^* \coloneqq \{b^* \mid b \in \mathfrak{B}\}$. With respect to this choice of basis, the dual $\phi^* \colon P^* \to P^*$ is represented by the transpose A^{t} of A. The complex $\mathrm{RHom}_R(C,R)[-1]$ is represented by $P^* \xrightarrow{\phi^*} P^*$ so that $\mathrm{coker}(\phi^*) = H^1(\mathrm{RHom}_R(C,R)[-1])$, hence we deduce that $\mathrm{Fitt}^j_R(H^1(\mathrm{RHom}_R(C,R)[-1]))$ is equal to the ideal of R generated by the $(n-j)\times(n-j)$-minors of $\iota(A^{\mathrm{t}})$. Since the sets of minors of A and A^{t} are in bijective correspondence, this proves $\mathrm{Fitt}^j_R(H^1(\mathrm{RHom}_R(C,R)[-1]))$ is equal to $\mathrm{Fitt}^j_R(H^1(C))$, and hence completes the proof of the first claim.
The second claim is then true since if R is self-injective, then $\mathrm{Hom}_R(-,R)$ is an exact functor and so $H^1(\mathrm{RHom}_R(C,R)[-1]) = H^0(C)^*$. □

(2.37) Lemma. *Set $r_Y \coloneqq \mathrm{rk}_R(Y)$. Then, if $r > 0$, the following claims are valid.*

(i) Each choice of ordered basis $b_\bullet$ of the (free) R-module Y gives rise to a canonical homomorphism of R-modules $\vartheta_{C,b_\bullet} \colon \mathrm{Det}_R(C) \to \bigcap^r_R H^0(C)$.

(ii) Let $R \to R'$ be a surjective homomorphism of rings, with R' as in § 2.1.2. Then $C' \coloneqq R' \otimes^{\mathbb{L}}_R C$ satisfies the conditions of Lemma 2.35 with R replaced by R'. In addition, $Y' \coloneqq R' \otimes_R Y$ is a free R'-module quotient of $H^1(C') \cong R' \otimes_R H^1(C)$ of rank r_Y, the image $b'_\bullet$ of the basis $b_\bullet$ in (i) is an ordered R'-basis of Y' and there exists a canonical commutative diagram of R-module homomorphisms

$$\begin{array}{ccc} \mathrm{Det}_R(C) & \xrightarrow{\vartheta_{C,b_\bullet}} & \bigcap^r_R H^0(C) \\ \downarrow{\scriptstyle \pi_{C,R'}} & & \downarrow{\scriptstyle \bigcap^r_R \pi_{C,R'}} \\ \mathrm{Det}_{R'}(C') & \xrightarrow{\vartheta_{C',b'_\bullet}} & \bigcap^r_{R'} H^0(C'). \end{array}$$

Here $\pi_{C,R'}$ is the canonical composite $\mathrm{Det}_R(C) \to R' \otimes_R \mathrm{Det}_R(C) \cong \mathrm{Det}_{R'}(C')$ and the map $\bigcap^r_R \pi_{C,R'}$ is defined in the course of the proof below.

Proof. Set $a \coloneqq \chi_R(C)$. Then, since $r = r_Y + a > 0$, the argument of Lemma 2.35 (ii) implies we can fix a natural number $n \geq r$ such that C has a resolution $P^\bullet$ of the form $R^{\oplus n} \xrightarrow{\phi} R^{\oplus(n-r)} \oplus Y$. Then $\mathrm{rk}_R(R^{\oplus(n-r)} \oplus Y) = n - a$ and we consider the composite homomorphism of R-modules

$$\begin{aligned} \vartheta_{P^\bullet,b_\bullet} \colon \mathrm{Det}_R(P^\bullet) &= \big(\textstyle\bigwedge^n_R R^{\oplus n}\big) \otimes_R \big(\textstyle\bigwedge^{n-a}_R (R^{\oplus(n-r)} \oplus Y)\big)^* \\ &\xrightarrow{\simeq} \big(\textstyle\bigwedge^n_R R^{\oplus n}\big) \otimes_R \big(\textstyle\bigwedge^{n-r}_R R^{\oplus(n-r)}\big)^* \otimes_R \big(\textstyle\bigwedge^{r_Y}_R Y\big)^* \\ &\xrightarrow{\simeq} \big(\textstyle\bigwedge^n_R R^{\oplus n}\big) \otimes_R \textstyle\bigwedge^{n-r}_R (R^{\oplus(n-r)})^* \\ &\xrightarrow{\vartheta_\phi} \textstyle\bigcap^r_R H^0(C). \end{aligned}$$

Here the first isomorphism is clear and the second is induced by the isomorphism $\big(\bigwedge^{r_Y}_R Y\big)^* \to R$ that evaluates elements on $\bigwedge^{i=r_Y}_{i=1} b_i$ and the canonical isomorphism $\big(\bigwedge^{n-r}_R R^{\oplus(n-r)}\big)^* \cong \bigwedge^{n-r}_R (R^{\oplus(n-r)})^*$. In addition, the map ϑ_ϕ is defined by the condition that, for all $a \in \bigwedge^n_R R^{\oplus n}$

and $f_i \in (R^{\oplus(n-r)})^*$, one has

$$\vartheta_\phi\big(a \otimes_R \wedge_{i\in[n-r]} f_i\big) = (-1)^{r(n-r)} \cdot \big(\wedge_{i\in[n-r]}(f_i \circ \phi)\big)(a) \in {\bigwedge}_R^r R^{\oplus n}.$$

In particular, writing $\{\tilde{b}_i\}_{i\in[n-r]}$ for the standard ordered basis of $R^{\oplus(n-r)}$, the fact that $\mathrm{im}(\vartheta_\phi) \subseteq \bigcap_R^r H^0(C)$ follows by applying Lemma 2.17 (b) to the exact sequence

$$0 \to H^0(C) \to R^n \xrightarrow{(\tilde{b}_i^* \circ \phi)_{i\in[n-r]}} \bigoplus_{i\in[n-r]} R$$

that is induced by the fixed identification $H^0(C) \cong \ker(\phi)$ (after noting that $\mathrm{im}(\phi)$ is contained in the direct summand $R^{\oplus(n-r)}$ of $R^{\oplus(n-r)} \oplus Y$).

To prove that this construction only depends on the pair $(C, b_\bullet)$, and hence complete the proof of (i), we now fix an alternative representative $\widetilde{P}^\bullet$ for C of the form $R^{\oplus n'} \xrightarrow{\phi'} R^{\oplus(n'-r)} \oplus Y$ and use it to construct a map $\vartheta_{\widetilde{P}^\bullet, b_\bullet}$ just as above. We can then fix a quasi-isomorphism $\theta^\bullet \colon \widetilde{P}^\bullet \to P^\bullet$ of complexes of R-modules with the property that $H^0(\theta^\bullet)$ and $H^1(\theta^\bullet)$ induce the identity maps on $H^0(C)$ and $H^1(C)$. Then, if necessary after replacing $\widetilde{P}^\bullet$ by its direct sum with a complex $R^{\oplus m} \xrightarrow{\mathrm{id}} R^{\oplus m}$ for a suitable natural number m, we can assume that the maps θ^0 and θ^1 are both surjective (without changing $\mathrm{Det}_R(\widetilde{P}^\bullet)$). Writing $\ker(\theta^\bullet)$ for the complex $\ker(\theta^0) \xrightarrow{\phi} \ker(\theta^1)$, we thereby obtain a short exact sequence of complexes of R-modules $0 \to \ker(\theta^\bullet) \to \widetilde{P}^\bullet \xrightarrow{\theta^\bullet} P^\bullet \to 0$. This sequence implies firstly that $\ker(\theta^\bullet)$ is acyclic and then, by taking determinants, induces an isomorphism of R-modules

$$\iota\colon\ \mathrm{Det}_R(\widetilde{P}^\bullet) \cong \mathrm{Det}_R(\ker(\theta^\bullet)) \otimes_R \mathrm{Det}_R(P^\bullet) \cong \mathrm{Det}_R(P^\bullet)$$

that is independent of the choice of $\theta^\bullet$. By an easy diagram chase, one then checks that $\vartheta_{\widetilde{P}^\bullet, b_\bullet} = \vartheta_{P^\bullet, b_\bullet} \circ \iota$. This equality implies that the above construction gives a well-defined map $\mathrm{Det}_R(C) \to \bigcap_R^r H^0(C)$ that only depends on C and the choice of basis $b_\bullet$, as required.

To prove (ii), we note that C' is represented by the complex $R' \otimes_R P^\bullet$. Given this representative, all assertions of (ii) are clear except for the commutative diagram. To construct this diagram, we note Lemma 2.17 (a) implies the existence of an exact commutative diagram of R-modules

$$\begin{array}{ccccccc} 0 \longrightarrow & \bigcap_R^r H^0(C) & \longrightarrow & \bigwedge_R^r P & \longrightarrow & P \otimes_R \bigwedge_R^{r-1} P \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \bigcap_{R'}^r H^0(C') & \longrightarrow & \bigwedge_{R'}^r P' & \longrightarrow & P' \otimes_{R'} \bigwedge_{R'}^{r-1} P'. \end{array}$$

Here we set $P = R^n \cong R^{n-r} \oplus Y$ and $P' := R' \otimes_R P$ and the solid vertical arrows are the natural projection maps. In particular, since the square in this diagram commutes, there exists a dashed arrow that makes the entire diagram commutative. It is easily checked that this dashed arrow is independent of the choice of representative $P^\bullet$ of C and we denote it by $\bigcap_R^r \pi_{C,R'}$. Given this explicit construction of $\bigcap_R^r \pi_{C,R'}$, and those of the maps $\vartheta_{P^\bullet, b_\bullet}$ and $\vartheta_{R'\otimes_R P^\bullet, b'_\bullet}$ in (i), it is then a straightforward exercise to check that the diagram given in (ii) commutes, as required. □

(2.38) Remark. For data C, $b_\bullet$ and r_Y $(= |b_\bullet|)$ as in Lemma 2.37, the map $\vartheta_{C,b_\bullet}$ has the following further properties.

(i) For any ordered basis $\tilde{b}_\bullet$ of Y, define a matrix $U = U_{\tilde{b}_\bullet, b_\bullet}$ in $\mathrm{GL}_{r_Y}(R)$ by the condition $\tilde{b}_\bullet = U \cdot b_\bullet$. Then $\bigwedge_{i\in[r_Y]} \tilde{b}_i = \det(U) \cdot \bigwedge_{i\in[r_Y]} b_i$ and hence, by directly comparing the explicit construction of $\vartheta_{C,\tilde{b}_\bullet}$ and $\vartheta_{C,b_\bullet}$ in Lemma 2.37 (i), one verifies that $\vartheta_{C,\tilde{b}_\bullet} = \det(U) \cdot \vartheta_{C,b_\bullet}$.

(ii) If both R is reduced and $\chi_R(C) = 0$, then an alternative description of the map $\vartheta_{C,b_\bullet}$ is presented in [27, Prop. A.11].

3. Selmer structures, complexes, and modules

3.1. Galois cohomology

In this preliminary section we recall the definition, and basic properties, of some relevant Galois cohomology complexes.

3.1.1. Definitions and conventions

Let R be a local complete Gorenstein ring with finite residue field of characteristic p as in condition (2.5). (The constructions in this section can more generally be made for a ring that satisfies the weaker condition $(*)$ in [44, § 1.4] but since it is sufficient for our purposes to work under assumption (2.5), we shall do so in order to streamline exposition.)
Let $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be the descending filtration of finite-index ideals of R from (2.6), which has the property that the natural map $R \to \varprojlim_{n\in\mathbb{N}}(R/\mathfrak{a}_n)$ is an isomorphism. The discrete topology on the finite rings $R/\mathfrak{a}_n$ then naturally induces a topology on R. More generally, for any finitely presented R-module M we have an isomorphism $M \to \varprojlim_{n\in\mathbb{N}}(M/\mathfrak{a}_n M)$ that allows us to endow M with the inverse limit topology induced by the discrete topology on the finite modules $M/\mathfrak{a}_n M$.
Let G be a topological group and let M be an $R[G]$-module. Following Nekovář [80, Def. 3.3.4] we call an $R[G]$-module M 'ind-admissible' if it is equal to the union $\bigcup_{N\in\mathscr{S}(M)} N$ with $\mathscr{S}(M)$ the set of all $R[G]$-submodules $N \subseteq M$ which are finitely presented as an R-module and on which the action of G is continuous. For every ind-admissible $R[G]$-module M, we then define the (inhomogenous) continuous cochains of degree $i \geq 0$ as

$$\mathscr{C}^i(G, M) \coloneqq \varinjlim_{N\in\mathscr{S}(M)} \mathrm{Maps}_{\mathrm{cont}}(G^{\oplus i}, N),$$

and write $\mathscr{C}^\bullet(G, M)$ for the associated complex of (inhomogenous) continuous cochains (see, for example, [82, Ch. I, § 2] for the definition of the differential for this complex). We write $\mathrm{R}\Gamma(G, M)$ for the object of $D(R)$ defined by the complex $\mathscr{C}^\bullet(G, M)$.
If F denotes a perfect field with algebraic closure F^c and absolute Galois group $G_F \coloneqq \mathrm{Gal}(F^c/F)$, then we set

$$\mathrm{R}\Gamma(F, M) \coloneqq \mathrm{R}\Gamma(G_F, M).$$

We also fix a number field k and an algebraic closure k^c of k. If F is an extension of k in k^c, we write Π_F for the set of all places of F and Π_F^∞ and Π_F^p for the subsets of Π_F comprising places that are respectively archimedean and p-adic. We set $\Pi_F^\mathbb{R} \coloneqq \{\mathfrak{q} \in \Pi_F^\infty : F_\mathfrak{q} = \mathbb{R}\}$ and $\Pi_F^\mathbb{C} \coloneqq \{\mathfrak{q} \in \Pi_F^\infty : F_\mathfrak{q} = \mathbb{C}\}$ (so $\Pi_F^\infty = \Pi_F^\mathbb{R} \cup \Pi_F^\mathbb{C}$), and will often work under the following additional assumption on the pair (k, p).

$$\text{If } p = 2, \text{ then } \Pi_k^\mathbb{R} = \varnothing. \tag{3.1}$$

Denote by $S_{\mathrm{ram}}(M) \subseteq \Pi_F$ the subset of all places for which (a choice of) the inertia subgroup acts non-trivially on the G_F-module M, and set

$$S(M) \coloneqq \Pi_F^\infty \cup \Pi_F^p \cup S_{\mathrm{ram}}(M).$$

If $S \subseteq \Pi_F$ is a subset that contains $S_{\mathrm{ram}}(M)$, then M is naturally acted upon by $G_{F,S} \coloneqq \mathrm{Gal}(F_S/F)$ with F_S the maximal extension of F unramified outside S_F, and we define

$$\mathrm{R}\Gamma(\mathcal{O}_{F,S}, M) \coloneqq \mathrm{R}\Gamma(G_{F,S}, M).$$

This notation is motivated by the fact that $\mathrm{R}\Gamma(F, M)$ and $\mathrm{R}\Gamma(\mathcal{O}_{F,S}, M)$ coincide with the étale cohomology complexes $\mathrm{R}\Gamma((\mathrm{Spec}\, F)_{\text{ét}}, M)$ and $\mathrm{R}\Gamma((\mathrm{Spec}\, \mathcal{O}_{F,S})_{\text{ét}}, M)$ if M defines an étale sheaf on the étale site of $\mathrm{Spec}\, F$ (resp. of $\mathrm{Spec}\, \mathcal{O}_{F,S}$).
In each degree i, we set $H^i(\Lambda, M) \coloneqq H^i(\mathrm{R}\Gamma(\Lambda, M))$ if Λ denotes either F or $\mathcal{O}_{F,S}$.
We furthermore define

$$\mathscr{C}_c^\bullet(G_{F,S}, M) \coloneqq \mathrm{cone}\Big(\mathscr{C}^\bullet(G_{F,S}, M) \xrightarrow{\iota} \bigoplus\nolimits_{v\in S} \mathscr{C}^\bullet(G_{F_v}, M)\Big)[-1],$$

where ι denotes the natural restriction map, and write $\mathrm{R}\Gamma_c(\mathcal{O}_{F,S}, M)$ for the corresponding object of $D(R)$ (which coincides with the complex of compact-support étale cohomology if M defines an étale sheaf on $\operatorname{Spec}\mathcal{O}_{F,S}$). In particular, in $D(R)$ one has an exact triangle

$$\mathrm{R}\Gamma_c(\mathcal{O}_{F,S}, M) \to \mathrm{R}\Gamma(\mathcal{O}_{F,S}, M) \xrightarrow{\iota} \bigoplus_{\mathfrak{q}} \mathrm{R}\Gamma(F_{\mathfrak{q}}, M) \to . \tag{3.2}$$

In each degree i, we set $H^i_c(\mathcal{O}_{F,S}, M) \coloneqq H^i(\mathrm{R}\Gamma(\mathcal{O}_{F,S}, M))$.
For a place $\mathfrak{q} \in \Pi_F \setminus \Pi_F^\infty$, we write $\kappa_{\mathfrak{q}}$ for its residue field and define a complex in $D(R)$

$$\mathrm{R}\Gamma_f(F_{\mathfrak{q}}, M) \coloneqq \mathrm{R}\Gamma(\kappa_{\mathfrak{q}}, M^{\mathrm{Gal}(F^c_{\mathfrak{q}}/F^{\mathrm{ur}}_{\mathfrak{q}})}),$$

where $F^{\mathrm{ur}}_{\mathfrak{q}}$ denotes the maximal unramified extension of $F_{\mathfrak{q}}$ in $F^c_{\mathfrak{q}}$. Then there exists a natural composite 'inflation' morphism in $D(R)$

$$\iota_{M,\mathfrak{q}} \colon \mathrm{R}\Gamma_f(F_{\mathfrak{q}}, M) \to \mathrm{R}\Gamma(F_{\mathfrak{q}}, M^{\mathrm{Gal}(F^c_{\mathfrak{q}}/F^{\mathrm{ur}}_{\mathfrak{q}})}) \to \mathrm{R}\Gamma(F_{\mathfrak{q}}, M).$$

3.1.2. Properties

The case that A is a projective R-module is of particular interest to us and we make much use of the following general result. For a finitely generated projective R-module P we write $[P]_R$ for its class in $K_0(R)$. We also fix an injective hull $E_R(\Bbbk)$ of $\Bbbk$ and write $(-)^\vee \coloneqq \operatorname{Hom}_R(-, E_R(\Bbbk))$ for the Matlis dual functor.

(3.3) Lemma. *Assume R satisfies condition (2.5), let F be a number field, and let A be a finite-rank free R-module endowed with a continuous action of G_F such that $S(A)$ is finite. If $S \subseteq \Pi_F$ is a finite set that contains $S(A)$, then the following claims are valid.*

(i) $\mathrm{R}\Gamma_c(\mathcal{O}_{F,S}, A)$ *belongs to* $D^{\mathrm{perf},0}_{[1,3]}(R)$.

(ii) If $\mathfrak{q} \in \Pi_F \setminus (\Pi_F^\infty \cup \Pi_F^p)$, *then* $\mathrm{R}\Gamma(F_{\mathfrak{q}}, A)$ *belongs to* $D^{\mathrm{perf},0}_{[0,2]}(R)$.

(iii) If $\mathfrak{q} \in \Pi_F^p$, *then* $\mathrm{R}\Gamma(F_{\mathfrak{q}}, A)$ *belongs to* $D^{\mathrm{perf}}_{[0,2]}(R)$ *and* $\chi_R\big(\mathrm{R}\Gamma(F_{\mathfrak{q}}, A)\big) = -[F_{\mathfrak{q}} : \mathbb{Q}_p] \cdot [A]_R$.

(iv) If $\mathfrak{q} \in \Pi_F \setminus S(A)$, *then* $\mathrm{R}\Gamma_f(F_{\mathfrak{q}}, A)$ *belongs to* $D^{\mathrm{perf},0}_{[0,1]}(R)$.

(v) If (3.1) is valid, then $\mathrm{R}\Gamma(\mathcal{O}_{F,S}, A)$ *belongs to* $D^{\mathrm{perf}}_{[0,2]}(R)$ *and* $\mathrm{R}\Gamma(F_{\mathfrak{q}}, A)$ *belongs to* $D^{\mathrm{perf}}_{[0,0]}(R)$ *for each* $\mathfrak{q} \in \Pi_F^\infty$.

In the remainder of the result, we let Φ denote any one of the functors that sends A to $\mathrm{R}\Gamma_c(\mathcal{O}_{F,S}, A)$, to $\mathrm{R}\Gamma(F_{\mathfrak{q}}, A)$ for $\mathfrak{q} \in \Pi_F \setminus \Pi_F^\infty$, to $\mathrm{R}\Gamma_f(F_{\mathfrak{q}}, A)$ for $\mathfrak{q} \in \Pi_F \setminus S(A)$, or, if (3.1) is valid, either to $\mathrm{R}\Gamma(\mathcal{O}_{F,S}, A)$ or to $\mathrm{R}\Gamma(F_{\mathfrak{q}}, A)$ for $\mathfrak{q} \in \Pi_F^\infty$.

(vi) For any morphism $R \to R'$ of rings satisfying (2.5), there exists a natural isomorphism $\Phi(A) \otimes^{\mathbb{L}}_R R' \cong \Phi(A \otimes_R R')$ in $D(R')$.

(vii) For any morphism $R \to R'$ of finite rings satisfying (2.5), there exists a natural isomorphism $\mathrm{RHom}_R(R', \Phi(A^\vee)) \cong \Phi(\operatorname{Hom}_R(R', A^\vee))$ in $D(R')$.

(viii) Lemma 2.29 gives rise to a well-defined object $\varprojlim_{n\in\mathbb{N}} \Phi(A/\mathfrak{a}_n A)$ of $D(R)$ for which there exists a natural isomorphism $\Phi(A) \to \varprojlim_{n\in\mathbb{N}} \Phi(A/\mathfrak{a}_n A)$ in $D(R)$.

Proof. Claims (i) – (vi) follow directly from the general results established by Flach in [39, § 4 and § 5] and by Fukaya and Kato in [44, Prop. 1.6.5].
We next prove claim (vii). Under the conditions of (vii), the functor Φ satisfies the assumptions of [39, Prop. 3.1] with the algebra $\mathscr{B}$ in loc. cit. taken to be $\mathbb{Z}_p[\mathrm{Aut}_{\mathbb{Z}_p}(A)]$. It follows that there exists a bounded complex $P^\bullet$ of finitely generated projective $\mathscr{B}$-modules such that $\Phi(A) \cong \mathrm{RHom}_{\mathscr{B}}(P^\bullet, A)$ in $D(R)$. By construction, $\mathrm{RHom}_{\mathscr{B}}(P^\bullet, A)$ is a complex of finitely generated projective (hence also injective because R is self-injective) R-modules and so by definition $\mathrm{RHom}_R(R', \Phi(A)) = \mathrm{RHom}_R(R', \mathrm{RHom}_{\mathscr{B}}(P^\bullet, A))$ in $D(R')$. To prove (vii), it is therefore enough to construct an isomorphism in $D(R')$

$$\mathrm{RHom}_R(R', \mathrm{RHom}_{\mathscr{B}}(P^\bullet, A)) \to \mathrm{RHom}_{\mathscr{B}}(P^\bullet, \operatorname{Hom}_R(R', A)). \tag{3.4}$$

For this, we note that for any finitely generated projective $\mathscr{B}$-module P, there exists a natural isomorphism of R'-modules

$$\mathrm{Hom}_R(R', \mathrm{Hom}_{\mathscr{B}}(P, A)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{B}}(P, \mathrm{Hom}_R(R', A)), \quad f \mapsto (a \mapsto (b \mapsto f(b)(a))).$$

The map (3.4) is then induced by this isomorphism in every degree.

Finally, to prove (viii), we note that the family of complexes $(\Phi(A/\mathfrak{a}_n A))_n$ satisfies the conditions (a) and (b) of Lemma 2.29. Condition (a) is satisfied as a consequence of the respective result of (i) – (v) and the existence of the required isomorphisms in (b) follows from (vi) with $(R \to R', A)$ taken to be the pairs $(R/\mathfrak{a}_{n+1} \to R/\mathfrak{a}_n, A/\mathfrak{a}_{n+1}A)$. The construction of Lemma 2.29 therefore gives a complex $\varprojlim_{n\in\mathbb{N}} \Phi(A/\mathfrak{a}_n A)$ in $D(R)$. In addition, since $\Phi(A)$ belongs to $D^{\mathrm{perf}}(R)$, we may fix a bounded complex $P^\bullet$ of finitely generated projective R-modules that is isomorphic to $\Phi(A)$ in $D(R)$. Then, for each n, the isomorphism in (vi) applied to $R \to R/\mathfrak{a}_n$ implies that $(R/\mathfrak{a}_n) \otimes_R P^\bullet$ is isomorphic to $\Phi(A/\mathfrak{a}_n A)$ in $D(R/\mathfrak{a}_n)$. One therefore obtains a composite isomorphism in $D(R)$

$$\Phi(A) \cong P^\bullet \cong \varprojlim_{n\in\mathbb{N}} ((R/\mathfrak{a}_n) \otimes_R P^\bullet) \cong \varprojlim_{n\in\mathbb{N}} \Phi(A/\mathfrak{a}_n A),$$

in which the second isomorphism results from the fact each R-module P^i is finitely generated projective and the third is a consequence of the uniqueness assertion in Lemma 2.29. □

To recall the relevant duality theorems for Galois cohomology, we write $\mathrm{R}\Gamma_{\mathrm{Tate}}(F_\mathfrak{q}, A)$ for the standard complex computing Tate cohomology of A over $G_{F_\mathfrak{q}}$ for $\mathfrak{q} \in \Pi_F^{\mathbb{R}}$, and then define an object $\widetilde{\mathrm{R}\Gamma}_c(\mathcal{O}_{F,S}, A)$ of $D(R)$ by the triangle

$$\widetilde{\mathrm{R}\Gamma}_c(\mathcal{O}_{F,S}, A) \to \mathrm{R}\Gamma(\mathcal{O}_{F,S}, A) \xrightarrow{\lambda_S(A)} \bigoplus\nolimits_{\mathfrak{q}\in S} \mathrm{R}\Gamma_{\mathrm{Tate}}(F_\mathfrak{q}, A) \to \cdot$$

with $\lambda_S(A)$ the natural localisation morphism.

(3.5) Proposition. *If R satisfies (2.5) and A is a finitely generated free R-module, then the following claims are valid.*

(i) For each $\mathfrak{q} \in \Pi_F \setminus \Pi_F^\infty$, there are canonical isomorphisms in $D(R)$

$$\mathrm{RHom}_R(\mathrm{R}\Gamma(F_\mathfrak{q}, A^*(1)), R)[-2] \cong \mathrm{R}\Gamma(F_\mathfrak{q}, A) \cong \mathrm{RHom}_R(\mathrm{R}\Gamma(F_\mathfrak{q}, A^\vee(1), R)), E_R(\Bbbk))[-2].$$

(ii) There exist canonical isomorphisms in $D(R)$

$$\mathrm{RHom}_R(\widetilde{\mathrm{R}\Gamma}_c(\mathcal{O}_{F,S}, A^*(1)), R)[-3] \cong \mathrm{R}\Gamma(\mathcal{O}_{F,S}, A) \cong \mathrm{RHom}_R(\widetilde{\mathrm{R}\Gamma}_c(\mathcal{O}_{F,S}, A^\vee(1)), E_R(\Bbbk))[-3].$$

Proof. Since R is assumed to be Gorenstein, it is its own dualising module (cf. [7, Th. 3.3.7 (a)]). Given this fact, the isomorphisms in (i) and (ii) are respectively proved in [80, Th. 5.2.6 and § 5.7.5] (see also [18, § 5.1, Lem. 12 b)] for the first isomorphism in (ii)). □

3.2. Selmer structures of Nekovář and of Mazur–Rubin

Throughout this section, we fix a ring R satisfying (2.5) and an ind-admissible $R[G_k]$-module A for which $S(A)$ is finite.

3.2.1. Nekovář–Selmer structures

(3.6) Definition. *A 'Nekovář(–Selmer) structure' $\mathscr{F}$ on A comprises*

- *a finite set $S(\mathscr{F})$ of places of k that contains $S(A)$, and*
- *for each $\mathfrak{q} \in \Pi_k$, a morphism in $D(R)$*

$$\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A) \xrightarrow{\theta_{\mathscr{F},\mathfrak{q}}} \mathrm{R}\Gamma(k_\mathfrak{q}, A),$$

with $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A) = \mathrm{R}\Gamma_f(k_\mathfrak{q}, A)$ and $\theta_{\mathscr{F},\mathfrak{q}} = \iota_{A,\mathfrak{q}}$ for all $\mathfrak{q} \in \Pi_k \setminus S(\mathscr{F})$. We write $\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A)$ for the mapping cone of $\theta_{\mathscr{F},\mathfrak{q}}$ and in each degree i set

$$H^i_{\mathscr{F}}(k_\mathfrak{q}, A) \coloneqq H^i\big(\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A)\big) \text{ and } H^i_{/\mathscr{F}}(k_\mathfrak{q}, A) \coloneqq H^i\big(\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A)\big).$$

(3.7) Definition. *The 'Selmer complex' associated to a Nekovář structure* $\mathscr{F}$ *on* A *is the object* $\mathrm{R}\Gamma_{\mathscr{F}}(k,A)$ *of* $D(R)$ *that is defined (up to isomorphism) via the existence of an exact triangle*

$$\mathrm{R}\Gamma_{\mathscr{F}}(k,A) \to \mathrm{R}\Gamma(\mathcal{O}_{k,S(\mathscr{F})},A) \oplus \bigoplus\nolimits_{\mathfrak{q}\in S(\mathscr{F})} \mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}},A) \xrightarrow{(\iota_{S(\mathscr{F})},(\theta_{\mathscr{F},\mathfrak{q}})_{\mathfrak{q}})} \bigoplus\nolimits_{\mathfrak{q}\in S(\mathscr{F})} \mathrm{R}\Gamma(k_{\mathfrak{q}},A) \to \cdot \tag{3.8}$$

in which $\iota_{\mathscr{F}}$ *is the diagonal localisation map. In each degree* i *we set*

$$H^i_{\mathscr{F}}(k,A) := H^i\big(\mathrm{R}\Gamma_{\mathscr{F}}(k,A)\big).$$

We also say that $\mathscr{F}$ *is 'perfect' if* $\mathrm{R}\Gamma_{\mathscr{F}}(k,A)$ *belongs to* $D^{\mathrm{perf}}(R)$.

(3.9) Remark. The definition of $\mathrm{R}\Gamma_{\mathscr{F}}(k,A)$ given in Definition 3.7 specifies it only up to non-unique isomorphism in $D(R)$. However, in all examples that are relevant to this article, the structure $\mathscr{F}$ is given by data on the level of complexes so that the associated Selmer complex can be explicitly defined via a mapping cone construction.

The Octahedral axiom implies that the exact triangle (3.8) is equivalent to an exact triangle

$$\mathrm{R}\Gamma_{\mathscr{F}}(k,A) \to \mathrm{R}\Gamma(\mathcal{O}_{k,S(\mathscr{F})},A) \xrightarrow{\iota'_{S(\mathscr{F})}} \bigoplus\nolimits_{\mathfrak{q}\in S(\mathscr{F})} \mathrm{R}\Gamma_{/\mathscr{F}}(k_{\mathfrak{q}},A) \to \cdot \tag{3.10}$$

in $D(R)$ in which $\iota'_{S(\mathscr{F})}$ is induced by $\iota_{S(\mathscr{F})}$. The following notion allows us to extend this observation.

(3.11) Definition. *Let* $\mathscr{F}$ *and* $\mathscr{F}'$ *be Nekovář structures on* A. *Then* $\mathscr{F}'$ *'refines'* $\mathscr{F}$, *written* $\mathscr{F}' \le \mathscr{F}$, *if for each* $\mathfrak{q} \in \Pi_k$ *there exists a morphism* $j_{\mathscr{F}',\mathscr{F},\mathfrak{q}}\colon \mathrm{R}\Gamma_{\mathscr{F}'}(k_{\mathfrak{q}},A) \to \mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}},A)$ *in* $D(R)$ *with the following properties:* $\theta_{\mathscr{F}',\mathfrak{q}} = \theta_{\mathscr{F},\mathfrak{q}} \circ j_{\mathscr{F}',\mathscr{F},\mathfrak{q}}$ *and, if* $\mathfrak{q} \notin S(\mathscr{F}') \cup S(\mathscr{F})$, *then* $j_{\mathscr{F}',\mathscr{F},\mathfrak{q}}$ *the identity morphism* $\mathrm{R}\Gamma_f(k_{\mathfrak{q}},A) \to \mathrm{R}\Gamma_f(k_{\mathfrak{q}},A)$. *If* $\mathscr{F}'$ *refines* $\mathscr{F}$, *then for each* $\mathfrak{q} \in \Pi_k$ *we write* $\mathrm{R}\Gamma_{\mathscr{F}/\mathscr{F}'}(k_{\mathfrak{q}},A)$ *for the mapping cone of the given morphism* $j_{\mathscr{F}',\mathscr{F},\mathfrak{q}}$.

(3.12) Lemma. *The following claims are valid for any Nekovář structure* $\mathscr{F}$ *on* A.

(i) If S *is any finite subset of* Π_k *with* $S(\mathscr{F}) \subseteq S$, *then there exists an exact triangle in* $D(R)$

$$\mathrm{R}\Gamma_{\mathscr{F}}(k,A) \to \mathrm{R}\Gamma(\mathcal{O}_{k,S},A) \oplus \bigoplus\nolimits_{\mathfrak{q}\in S} \mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}},A) \xrightarrow{(\iota_S,(\theta_{\mathscr{F},\mathfrak{q}})_{\mathfrak{q}})} \bigoplus\nolimits_{\mathfrak{q}\in S} \mathrm{R}\Gamma(k_{\mathfrak{q}},A) \to \cdot$$

in which ι_S *is the diagonal localisation map*

(ii) Let $\mathscr{F}'$ *be a refinement of* $\mathscr{F}$. *Then there exists a canonical exact triangle in* $D(R)$

$$\mathrm{R}\Gamma_{\mathscr{F}'}(k,A) \to \mathrm{R}\Gamma_{\mathscr{F}}(k,A) \to \bigoplus\nolimits_{\mathfrak{q}\in S(\mathscr{F}')\cup S(\mathscr{F})} \mathrm{R}\Gamma_{\mathscr{F}/\mathscr{F}'}(k_{\mathfrak{q}},A) \to \cdot$$

(iii) If A *is a finite-rank free* R*-module, then* $\mathscr{F}$ *is perfect if and only if, for every* $\mathfrak{q}$ *in* $S(\mathscr{F})$, *the complex* $\mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}},A)$ *belongs to* $D^{\mathrm{perf}}(R)$. *If this is the case, then in* $K_0(R)$ *one has*

$$\chi_R\big(\mathrm{R}\Gamma_{\mathscr{F}}(k,A)\big) = \sum\nolimits_{\mathfrak{q}\in S(\mathscr{F})} \chi_R\big(\mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}},A)\big).$$

(iv) If either $i<0$ *or* $i>3$, *then* $H^i_{\mathscr{F}}(k,A) = \bigoplus_{\mathfrak{q}\in S(\mathscr{F})} H^i_{\mathscr{F}}(k_{\mathfrak{q}},A)$. *The same equality is valid for* $i=3$ *if* A *is a finitely generated free* R*-module and there exists a place* $\mathfrak{q}_0 \in S(\mathscr{F})\setminus\Pi_k^{\mathbb{R}}$ *for which* $H^2(\theta_{\mathscr{F},\mathfrak{q}_0})$ *is bijective.*

Proof. For a finite subset Σ of Π_k we write $C_\Sigma(A), C_\Sigma^f(A)$ and $C_\Sigma^{\mathscr{F}}(A)$ for the respective direct sums over $\mathfrak{q} \in \Sigma$ of the complexes $\mathrm{R}\Gamma(k_{\mathfrak{q}},A), \mathrm{R}\Gamma_f(k_{\mathfrak{q}},A)$ and $\mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}},A)$. We then recall that, for any finite subsets S' and S with $S(A) \subseteq S' \subseteq S$, there exists a canonical 'inflation-restriction' exact triangle in $D(R)$

$$\mathrm{R}\Gamma(\mathcal{O}_{k,S'},A) \xrightarrow{\iota_{S',S}(A)} \mathrm{R}\Gamma(\mathcal{O}_{k,S},A) \oplus C^f_{S\setminus S'}(A) \xrightarrow{(\iota'_{S',S},(\iota_{\mathfrak{q},A})_{\mathfrak{q}})} C_{S\setminus S'}(A) \to \tag{3.13}$$

in which $\iota'_{S',S}$ is the canonical localisation morphism.

To prove (i) we assume $S' = S(\mathscr{F})$ and consider the following diagram in $D(R)$

$$\begin{array}{ccccccc} \mathrm{R\Gamma}(\mathcal{O}_{k,S'},A)\oplus C^{\mathscr{F}}_{S'}(A) & \longrightarrow & \big(\mathrm{R\Gamma}(\mathcal{O}_{k,S},A)\oplus C^{f}_{S\setminus S'}(A)\big)\oplus C^{\mathscr{F}}_{S'}(A) & \longrightarrow & C_{S\setminus S'}(A) & \longrightarrow & . \\ \Big\downarrow{\scriptstyle (\iota_{\mathscr{F}},(\theta_{\mathscr{F},\mathfrak{q}})_{\mathfrak{q}})} & & \Big\downarrow{\scriptstyle ((\iota_S,(\iota_{\mathfrak{q},A})_{\mathfrak{q}}),(\theta_{\mathscr{F},\mathfrak{q}})_{\mathfrak{q}})} & & \Big\| & & \\ C_{S'}(A) & \longrightarrow & C_S(A) & \longrightarrow & C_{S\setminus S'}(A) & \longrightarrow & . \end{array}$$

Here the upper row is the exact triangle induced by (3.13) and the lower row is the canonical exact triangle. In addition, because $C^{f}_{S\setminus S'}(A) = C^{\mathscr{F}}_{S\setminus S'}(A)$ (since $S' = S(\mathscr{F})$), the central vertical map agrees with $(\iota_S,(\theta_{\mathscr{F},\mathfrak{q}}))_{\mathfrak{q}})$ where $\mathfrak{q}$ runs over S, and the diagram commutes. The Octahedral axiom therefore implies that the mapping fibres of the first and second vertical morphisms are canonically isomorphic in $D(R)$ and this implies (i).
To prove (ii) we set $S := S(\mathscr{F}') \cup S(\mathscr{F})$ and consider the following diagram in $D(R)$

$$\begin{array}{ccccccc} \mathrm{R\Gamma}_{\mathscr{F}'}(k,A) & \longrightarrow & \mathrm{R\Gamma}(\mathcal{O}_{k,S},A)\oplus C^{\mathscr{F}'}_{S}(A) & \xrightarrow{(\iota_S,(\theta_{\mathscr{F}',\mathfrak{q}})_{\mathfrak{q}})} & C_S(A) & \longrightarrow & . \\ & & \Big\downarrow{\scriptstyle (\mathrm{id},(j_{\mathscr{F}',\mathscr{F},\mathfrak{q}})_{\mathfrak{q}})} & & \Big\| & & \\ \mathrm{R\Gamma}_{\mathscr{F}}(k,A) & \longrightarrow & \mathrm{R\Gamma}(\mathcal{O}_{k,S},A)\oplus C^{\mathscr{F}}_{S}(A) & \xrightarrow{(\iota_S,(\theta_{\mathscr{F},\mathfrak{q}})_{\mathfrak{q}})} & C_S(A) & \longrightarrow & . \end{array}$$

Here the two rows are the exact triangles obtained from (i) and the square commutes by choice of the morphisms $j_{\mathscr{F}',\mathscr{F},\mathfrak{q}}$. In particular, the Octahedral axiom implies both that the diagram can be completed to give a morphism of exact triangles and hence that there exists an exact triangle as stated in (ii).
We note next that the exact triangle (3.2) defining $\mathrm{R\Gamma}_c(\mathcal{O}_{k,S},A)$ combines with the exact triangle in (i) (with $S = S(\mathscr{F})$) to imply, via the Octahedral axiom, the existence of an exact triangle in $D(R)$

$$\mathrm{R\Gamma}_c(\mathcal{O}_{k,S},A) \to \mathrm{R\Gamma}_{\mathscr{F}}(k,A) \to \bigoplus\nolimits_{\mathfrak{q}\in S}\mathrm{R\Gamma}_{\mathscr{F}}(k_{\mathfrak{q}},A) \to . \tag{3.14}$$

Given this triangle, claim (iii) and the first assertion of (iv) both follow directly from the result of Lemma 3.3 (i). To prove the second assertion of (iv) we note that the long exact cohomology sequence of the last displayed triangle gives an exact sequence

$$\bigoplus\nolimits_{\mathfrak{q}\in S}H^2_{\mathscr{F}}(k_{\mathfrak{q}},A) \xrightarrow{\gamma} H^3_c(\mathcal{O}_{k,S},A) \to H^3_{\mathscr{F}}(k,A) \to \bigoplus\nolimits_{\mathfrak{q}\in S}H^3_{\mathscr{F}}(k_{\mathfrak{q}},A) \to H^4_c(\mathcal{O}_{k,S},A) = (0).$$

It is therefore enough to show that the stated hypothesis implies γ is surjective. This is true, since the compatibility of local and global duality (as respectively expressed by the second isomorphisms in Proposition 3.5 (i) and (ii)) identifies the composite map

$$H^2(k_{\mathfrak{q}_0},A) \xrightarrow{H^2(\theta_{\mathscr{F},\mathfrak{q}_0})^{-1}} H^2_{\mathscr{F}}(k_{\mathfrak{q}_0},A) \subseteq \bigoplus\nolimits_{\mathfrak{q}\in S}H^2_{\mathscr{F}}(k_{\mathfrak{q}},A) \xrightarrow{\gamma} H^3_c(\mathcal{O}_{k,S},A)$$

with the Matlis-dual of the inclusion map $H^0(\mathcal{O}_{k,S},A^\vee(1)) \to H^0(k_{\mathfrak{q}_0},A^\vee(1))$. Since the latter map is injective, its Matlis duals is surjective, as claimed. □

(3.15) Example. (Greenberg–Nekovář structures) Fix a finite subset S of Π_k with $S(A) \subseteq S$ and assume to be given, for $\mathfrak{q} \in S$, an ind-admissible $R[G_{k_\mathfrak{q}}]$-module $A_\mathfrak{q}$ for which there exists a morphism $j_\mathfrak{q}\colon A_\mathfrak{q} \to A$ of $R[G_{k_\mathfrak{q}}]$-modules. Then one obtains a Nekovář structure $\mathscr{F} = \mathscr{F}((A_\mathfrak{q},j_\mathfrak{q})_{\mathfrak{q}\in S})$ with $S(\mathscr{F}) := S$ by setting $\mathrm{R\Gamma}_{\mathscr{F}}(k_\mathfrak{q},A) := \mathrm{R\Gamma}(k_\mathfrak{q},A_\mathfrak{q})$ and $\theta_{\mathcal{F},\mathfrak{q}} := \mathrm{R\Gamma}(k_\mathfrak{q},j_\mathfrak{q})$ for every $\mathfrak{q} \in S$. This approach is discussed more fully in [80, § 7.8], and the following examples will play a key role in subsequent arguments.

(i) The 'relaxed' Nekovář structure $\mathscr{F}_{\mathrm{rel}} = \mathscr{F}_{\mathrm{rel}}(A,S)$ for A and S. One has $S(\mathscr{F}_{\mathrm{rel}}) = S$ and, for each $\mathfrak{q} \in S$, sets

$$\big(\mathrm{R\Gamma}_{\mathscr{F}_{\mathrm{rel}}}(k_\mathfrak{q},A),\theta_{\mathscr{F}_{\mathrm{rel}},\mathfrak{q}}\big) := \big(\mathrm{R\Gamma}(k_\mathfrak{q},A),\mathrm{id}\big).$$

Hence $\mathrm{R\Gamma}_{\mathscr{F}_{\mathrm{rel}}}(k,A) = \mathrm{R\Gamma}(\mathcal{O}_{k,S},A)$ and so Lemma 3.3 (v) implies that $\mathscr{F}_{\mathrm{rel}}$ is perfect if A is a free R-module of finite rank and (3.1) is satisfied.

(ii) The 'canonical' Nekovář structure $\mathscr{F}_{\rm can} = \mathscr{F}_{\rm can}(A,S)$ for A and S refines $\mathscr{F}_{\rm rel}(A,S)$. One has $S(\mathscr{F}_{\rm can}) = S$ and for $\mathfrak{q} \in S$ sets

$$\big(\mathrm{R}\Gamma_{\mathscr{F}_{\rm can}}(k_\mathfrak{q},A),\theta_{\mathscr{F}_{\rm can},\mathfrak{q}}\big) := \begin{cases}\big(\mathrm{R}\Gamma_f(k_\mathfrak{q},A),\iota_{A,\mathfrak{q}}\big), & \text{if } \mathfrak{q}\in S\setminus(\Pi_k^\infty\cup\Pi_k^p)\\ \big(\mathrm{R}\Gamma(k_\mathfrak{q},A),\mathrm{id}\big), & \text{if } \mathfrak{q}\in \Pi_k^\infty\cup\Pi_k^p.\end{cases}$$

In particular, if A is a free R-module of finite rank, then Lemma 3.12 (iii) combines with Lemma 3.3 (ii) and (iv) to imply $\mathscr{F}_{\rm can}$ is perfect if and only if, for each $\mathfrak{q} \in \Pi_k^\infty$, the $G_{k_\mathfrak{q}}$-module A is cohomologically-trivial and the R-module $H^0(k_\mathfrak{q},A)$ is projective and, for each $\mathfrak{q}\in S(A)\setminus(\Pi_k^\infty\cup\Pi_k^p)$, the complex $\mathrm{R}\Gamma_f(k_\mathfrak{q},A)$ belongs to $D^{\rm perf}(R)$.

(iii) The 'strict' Nekovář structure $\mathscr{F}_{\rm str} = \mathscr{F}_{\rm str}(A,S)$ for A and S refines $\mathscr{F}_{\rm can}(A,S)$, and hence also $\mathscr{F}_{\rm rel}(A,S)$. One has $S(\mathscr{F}_{\rm str}) = S$ and for $\mathfrak{q}\in S$ sets

$$\big(\mathrm{R}\Gamma_{\mathscr{F}_{\rm str}}(k_\mathfrak{q},A),\theta_{\mathscr{F}_{\rm str},\mathfrak{q}}\big) := (0,0).$$

One therefore has $\mathrm{R}\Gamma_{\mathscr{F}_{\rm str}}(k,A) = \mathrm{R}\Gamma_c(\mathcal{O}_{k,S},A)$ and so Lemma 3.3 (i) implies $\mathscr{F}_{\rm str}$ is perfect if A is a free R-module of finite rank.

The approach of Example 3.15 does not encompass all Nekovář structures of arithmetic interest. For instance, if S is a finite subset of Π_k with $S(A)\subseteq S$, then the specification for each $\mathfrak{q}\in S$ of a projective R-submodule $X_\mathfrak{q}$ of $H^1(k_\mathfrak{q},A)$ determines a perfect Nekovář structure $\mathscr{F} = \mathscr{F}((X_\mathfrak{q})_\mathfrak{q})$ with $S(\mathscr{F}) = S$ and, for each $\mathfrak{q}\in S$, $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q},A) = X_\mathfrak{q}[-1]$ and $\theta_{\mathscr{F},\mathfrak{q}}$ the unique morphism $X_\mathfrak{q}[-1]\to \mathrm{R}\Gamma(k_\mathfrak{q},A)$ in $D(R)$ for which $H^1(\theta_{\mathscr{F},\mathfrak{q}})$ is the inclusion $X_\mathfrak{q}\subseteq H^1(k_\mathfrak{q},A)$. In particular, this construction incorporates the 'perfect Selmer structures' introduced in [24, § 2] in order to formulate refined versions of the Birch and Swinnerton-Dyer conjecture for abelian varieties over number fields.
There are also several natural ways in which a given Nekovář structure $\mathscr{F}$ on A gives rise to further structures. In § 3.3 we will discuss natural notions of 'dual' Nekovář structure. To end this section, we record several other constructions that are also important for our approach.

(3.16) Example. (Σ-modifications) Let Σ be a finite subset of Π_k. Then the 'Σ-modification' $\mathscr{F}_\Sigma$ and 'Σ-comodification' $\mathscr{F}^\Sigma$ of $\mathscr{F}$ are the Nekovář structures on A that are specified as follows: $S(\mathscr{F}_\Sigma) = S(\mathscr{F}^\Sigma) := S(\mathscr{F})\cup\Sigma$; for $\mathfrak{q}\in S(\mathscr{F})$, one has

$$(\mathrm{R}\Gamma_{\mathscr{F}_\Sigma}(k_\mathfrak{q},A),\theta_{\mathscr{F}_\Sigma,\mathfrak{q}}) = (\mathrm{R}\Gamma_{\mathscr{F}^\Sigma}(k_\mathfrak{q},A),\theta_{\mathscr{F}^\Sigma,\mathfrak{q}}) := (\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q},A),\theta_{\mathscr{F},\mathfrak{q}});$$

for $\mathfrak{q}\in\Sigma\setminus S(\mathscr{F})$, one has

$$(\mathrm{R}\Gamma_{\mathscr{F}_\Sigma}(k_\mathfrak{q},A),\theta_{\mathscr{F}_\Sigma,\mathfrak{q}}) := (0,0) \quad\text{and}\quad (\mathrm{R}\Gamma_{\mathscr{F}^\Sigma}(k_\mathfrak{q},A),\theta_{\mathscr{F}^\Sigma,\mathfrak{q}}) := (\mathrm{R}\Gamma(k_\mathfrak{q},A),\mathrm{id}).$$

If $\Sigma=\varnothing$, then $\mathscr{F}_\Sigma = \mathscr{F}^\Sigma = \mathscr{F}$. In general, $\mathscr{F}_\Sigma$ refines $\mathscr{F}$ and Lemma 3.12 (ii) gives an exact triangle in $D(R)$

$$\mathrm{R}\Gamma_{\mathscr{F}_\Sigma}(k,A)\to\mathrm{R}\Gamma_{\mathscr{F}}(k,A)\to\bigoplus\nolimits_{\mathfrak{q}\in\Sigma\setminus S(\mathscr{F})}\mathrm{R}\Gamma_f(k_\mathfrak{q},A)\to\cdot \tag{3.17}$$

In addition, by comparing the exact triangles in Lemma 3.12 (i) for $\mathscr{F}$ and $\mathscr{F}^\Sigma$ (and with S taken to be $S(\mathscr{F})\cup\Sigma$ in both cases), one finds that local Tate duality (Theorem 3.5 (i)) for each $\mathfrak{q}\in\Sigma\setminus S(\mathscr{F})$ gives rise to an exact triangle in $D(R)$

$$\mathrm{R}\Gamma_{\mathscr{F}}(k,A)\to\mathrm{R}\Gamma_{\mathscr{F}^\Sigma}(k,A)\to\bigoplus\nolimits_{\mathfrak{q}\in\Sigma\setminus S(\mathscr{F})}\mathrm{R}\Gamma_f(k_\mathfrak{q},A^\vee(1))^\vee[-2]\to\cdot$$

These two displayed exact triangles combine with Lemma 3.3 (ii) to imply that $\mathscr{F}$ is perfect if and only if $\mathscr{F}_\Sigma$ and $\mathscr{F}^\Sigma$ are both perfect. As a concrete example, if $\mathscr{F}$ is the relaxed structure $\mathscr{F}_{\rm rel} = \mathscr{F}_{\rm rel}(A,S)$ from Example 3.15 (i), then one has $\mathrm{R}\Gamma_{\mathscr{F}_{\rm rel}}(k,A) = \mathrm{R}\Gamma(\mathcal{O}_{k,S},A)$. In particular, if $\Sigma\cap S=\varnothing$, then the exact triangle (3.17) implies that $\mathrm{R}\Gamma_{\mathscr{F}_{\rm rel,\Sigma}}(k,A)$ coincides with the 'Σ-modified étale cohomology' complexes $\mathrm{R}\Gamma_\Sigma(\mathcal{O}_{k,S},A)$ used in [26] and [27]. In the case $A=\mathbb{Z}_p(1)$ such constructions were first used by Gross [47] and Rubin [91] in the context of Stark's conjectures (see also [92, Ch. IX, § 5]).

(3.18) Example. (Induced structures)

(i) If $\iota\colon A' \to A$ is a homomorphism of continuous $R[G_k]$-modules, then $\mathscr{F}$ induces a Nekovář structure $\iota^*(\mathscr{F})$ on A' as follows. One has $S(\iota^*(\mathscr{F})) := S(\mathscr{F})$; for $\mathfrak{q} \in S(\mathscr{F})$ one defines $\mathrm{R\Gamma}_{\iota^*(\mathscr{F})}(k_\mathfrak{q}, A')$ via the exact triangle in $D(R)$

$$\mathrm{R\Gamma}_{\iota^*(\mathscr{F})}(k_\mathfrak{q}, A') \to \mathrm{R\Gamma}(k_\mathfrak{q}, A') \oplus \mathrm{R\Gamma}_{\mathscr{F}}(k_\mathfrak{q}, A) \xrightarrow{(\mathrm{R\Gamma}(k_\mathfrak{q},\iota),\theta_{\mathscr{F},\mathfrak{q}})} \mathrm{R\Gamma}(k_\mathfrak{q}, A) \to,$$

and takes $\theta_{\iota^*(\mathscr{F}),\mathfrak{q}}$ to be the morphism induced by the first morphism in this triangle. When the map ι is clear from context, we will write $\mathscr{F}_{A'}$ in place of $\iota^*(\mathscr{F})$.

(ii) If $j\colon A \to A'$ is a homomorphism of continuous $R[G_k]$-modules, then $\mathscr{F}$ induces a Nekovář structure $j_*(\mathscr{F})$ on A' as follows. One has $S(j_*(\mathscr{F})) := S(\mathscr{F})$; for $\mathfrak{q} \in S(\mathscr{F})$, one defines $\mathrm{R\Gamma}_{j_*(\mathscr{F})}(k_\mathfrak{q}, A')$ to be $\mathrm{R\Gamma}_{\mathscr{F}}(k_\mathfrak{q}, A)$ and $\theta_{j_*(\mathscr{F}),\mathfrak{q}}$ to be the composite $\mathrm{R\Gamma}(k_\mathfrak{q}, j)\circ\theta_{\mathscr{F},\mathfrak{q}}$. When the map ι is clear from context, we will write $\mathscr{F}_{A'}$ in place of $j_*(\mathscr{F})$.

(iii) Fix a morphism of rings $R \to R'$ satisfying (2.5). Assume A is a finitely generated free R-module and that condition (3.1) is valid. Then, for every place $\mathfrak{q} \in \Pi_k$, there exists a natural isomorphism $\mathrm{R\Gamma}(k_\mathfrak{q}, A) \otimes^{\mathbb{L}}_R R' \cong \mathrm{R\Gamma}(k_\mathfrak{q}, A \otimes_R R')$ in $D(R')$ (cf. Lemma 3.3 (vi)) and so one can specify a Nekovář structure $\mathscr{F} \otimes_R R'$ on $A \otimes_R R'$ as follows. One has $S(\mathscr{F} \otimes_R R') := S(\mathscr{F})$; for $\mathfrak{q} \in S(\mathscr{F})$, one sets

$$\mathrm{R\Gamma}_{\mathscr{F}\otimes_R R'}(k_\mathfrak{q}, A \otimes_R R') := \mathrm{R\Gamma}_{\mathscr{F}}(k_\mathfrak{q}, A) \otimes^{\mathbb{L}}_R R'$$

and takes $\theta_{\mathscr{F}\otimes_R R',\mathfrak{q}}$ to be the following composite morphism in $D(R')$

$$\mathrm{R\Gamma}_{\mathscr{F}\otimes_R R'}(k_\mathfrak{q}, A \otimes_R R') \xrightarrow{\theta_{\mathscr{F},\mathfrak{q}}\otimes^{\mathbb{L}}_R R'} \mathrm{R\Gamma}(k_\mathfrak{q}, A) \otimes^{\mathbb{L}}_R R' \cong \mathrm{R\Gamma}(k_\mathfrak{q}, A \otimes_R R').$$

(iv) Assume the same conditions as in (iii) and fix finite subsets S and Σ of Π_k with $S(A) \subseteq S$ and $S \cap \Sigma = \varnothing$. Then, for the structures defined in Examples 3.15 and 3.16, Lemma 3.3 (vi) implies that the corresponding induced structures $\mathscr{F}_{\mathrm{rel}}(A,S)_\Sigma \otimes_R R'$, $\mathscr{F}_{\mathrm{rel}}(A,S)^\Sigma \otimes_R R'$, $\mathscr{F}_{\mathrm{str}}(A,S)_\Sigma \otimes_R R'$ and $\mathscr{F}_{\mathrm{str}}(A,S)^\Sigma \otimes_R R'$ respectively identify with $\mathscr{F}_{\mathrm{rel}}(A \otimes_R R', S)_\Sigma$, $\mathscr{F}_{\mathrm{rel}}(A \otimes_R R', S)^\Sigma$, $\mathscr{F}_{\mathrm{str}}(A \otimes_R R', S)_\Sigma$ and $\mathscr{F}_{\mathrm{str}}(A \otimes_R R', S)^\Sigma$.

For the induced structures defined in the last example, one has the following useful 'control theorems' for Selmer complexes.

(3.19) Proposition. *Assume condition (3.1) is valid, and let $\mathscr{F}$ be a Nekovář structure on a finitely generated free R-module A. Then the following claims are valid.*

(i) For every morphism $R \to R'$ of rings satisfying (2.5) one has a natural isomorphism $\mathrm{R\Gamma}_{\mathscr{F}}(k, A) \otimes^{\mathbb{L}}_R R' \cong \mathrm{R\Gamma}_{\mathscr{F}\otimes_R R'}(k, A \otimes_R R')$ in $D(R')$.

(ii) Let R be a finite ring satisfying (2.5) and $I \subseteq R$ an ideal. Then one has a natural isomorphism $\mathrm{RHom}_R(R/I, \mathrm{R\Gamma}_{\mathscr{F}}(k, A)) \cong \mathrm{R\Gamma}_{\mathscr{F}_{A[I]}}(k, A[I])$ in $D(R/I)$.

(iii) Assume $\mathscr{F}$ is perfect and, for $n \in \mathbb{N}$, set $A_n := A \otimes_R (R/\mathfrak{a}_n)$ and $\mathscr{F}_n := \mathscr{F} \otimes_R (R/\mathfrak{a}_n)$. Then Lemma 2.29 defines an object $\varprojlim_{n\in\mathbb{N}} \mathrm{R\Gamma}_{\mathscr{F}_n}(k, A_n)$ of $D^{\mathrm{perf}}(R)$ that is naturally isomorphic to $\mathrm{R\Gamma}_{\mathscr{F}}(k, A)$. Hence, in each degree i, one has $H^i_{\mathscr{F}}(k, A) = \varprojlim_{n\in\mathbb{N}} H^i_{\mathscr{F}_n}(k, A_n)$.

Proof. The mapping cone of a morphism in $D(R')$ is unique up to ismorphism. Hence, given the explicit definition of each complex $\mathrm{R\Gamma}_{\mathscr{F}\otimes_R R'}(k_\mathfrak{q}, A \otimes_R R')$ and morphism $\theta_{\mathscr{F}_{A\otimes_R R'},\mathfrak{q}}$, the isomorphism in (i) results directly by comparing the triangles (3.14) for the structures $\mathscr{F}$ and $\mathscr{F}_{A\otimes_R R'}$ and taking account of the isomorphisms in Lemma 3.3 (vi).

The isomorphism in (ii) is obtained by a similar application of Lemma 3.3 (vii) with $R' = R/I$. To prove (iii), the fact $\mathscr{F}$ is perfect allows us to fix a bounded complex $P^\bullet$ of finitely generated projective R-modules that is isomorphic in $D(R)$ to $\mathrm{R\Gamma}_{\mathscr{F}}(k, A)$. We then fix integers a and b with $a < b$ and such that $P^i = (0)$ if either $i < a$ or $i > b$. Then by applying the isomorphism in (i) to the morphism $R \to R/\mathfrak{a}_n$, respectively $R/\mathfrak{a}_{n+1} \to R/\mathfrak{a}_n$, we deduce $\mathrm{R\Gamma}_{\mathscr{F}_n}(k, A/\mathfrak{a}_n A)$ belongs to $D^{\mathrm{perf}}_{[a,b]}(R/\mathfrak{a}_n)$, respectively there exists an isomorphism $\mathrm{R\Gamma}_{\mathscr{F}_{n+1}}(k, A_{n+1}) \otimes^{\mathbb{L}}_{R/\mathfrak{a}_{n+1}}$

$R/\mathfrak{a}_n \cong \mathrm{R}\Gamma_{\mathscr{F}_n}(k, A_n)$ in $D(R/\mathfrak{a}_n)$. The hypotheses of Lemma 2.29 are therefore satisfied in this case so that $\varprojlim_{n\in\mathbb{N}} \mathrm{R}\Gamma_{\mathscr{F}_n}(k, A_n)$ is well-defined and isomorphic in $D(R)$ to $\mathrm{R}\Gamma_{\mathscr{F}}(k, A) \cong P^\bullet$ by the uniqueness assertion of the latter result. This isomorphism then induces an identification $H^i_{\mathscr{F}}(k, A) = \varprojlim_{n\in\mathbb{N}} H^i_{\mathscr{F}_n}(k, A_n)$ in each degree i since all modules $P^j/\mathfrak{a}_n$ and $H^j_{\mathscr{F}_n}(k, A_n)$ are finite and inverse limits are exact on the category of finite abelian groups. □

(3.20) Remark. The isomorphisms in Proposition 3.19 (i) and (ii) respectively give rise to convergent spectral sequences

$$E_2^{i,j} = \mathrm{Tor}^R_{-i}(H^j_{\mathscr{F}}(k, A), R') \quad \Rightarrow \quad E^{i+j} = H^{i+j}_{\mathscr{F}_{A\otimes_R R'}}(k, A \otimes_R R'),$$
$$E_2^{i,j} = \mathrm{Ext}^i_R(R/I, H^j_{\mathscr{F}}(k, A)) \quad \Rightarrow \quad E^{i+j} = H^{i+j}_{\mathscr{F}_{A[I]}}(k, A[I]).$$

In particular, if $H^0_{\mathscr{F}}(k, A)$ vanishes, then the latter spectral sequence induces an isomorphism $H^1_{\mathscr{F}}(k, A)[I] \cong H^1_{\mathscr{F}_{A[I]}}(k, A[I])$. This observation is the natural analogue for Nekovář structures of the result of [26, Cor. 3.8].

3.2.2. Mazur–Rubin–Selmer structures

For each place $\mathfrak{q} \in \Pi_k \setminus \Pi_k^\infty$, we now set

$$H^1_f(k_\mathfrak{q}, A) := H^1(\mathrm{R}\Gamma_f(k_\mathfrak{q}, A)) = \ker(H^1(k_\mathfrak{q}, A) \to H^1(k_\mathfrak{q}^{\mathrm{ur}}, A)).$$

(3.21) Definition. *A 'Mazur–Rubin(–Selmer) structure' $\mathcal{F}$ on A comprises*

- *a finite set $S(\mathcal{F})$ of places of k that contains $S(A)$, and*
- *for each $\mathfrak{q} \in \Pi_k$, an R-submodule*
$$H^1_{\mathcal{F}}(k_\mathfrak{q}, A) \subseteq H^1(k_\mathfrak{q}, A)$$
of $H^1(k_\mathfrak{q}, A)$, with $H^1_{\mathcal{F}}(k_\mathfrak{q}, A) := H^1_f(k_\mathfrak{q}, A)$ for all $\mathfrak{q} \in \Pi_k \setminus S(\mathcal{F})$.

A Mazur–Rubin structure $\mathcal{F}'$ on A 'refines' $\mathcal{F}$, written $\mathcal{F}' \leq \mathcal{F}$, if $S(\mathcal{F}) \subseteq S(\mathcal{F}')$ and for each $\mathfrak{q} \in \Pi_k$ one has $H^1_{\mathcal{F}'}(k_\mathfrak{q}, A) \subseteq H^1_{\mathcal{F}}(k_\mathfrak{q}, A)$.

The link between these structures and Nekovář structures is as follows.

(3.22) Lemma. *The following claims are valid.*

(i) A Nekovář structure $\mathscr{F}$ on A induces a canonical Mazur–Rubin structure $h(\mathscr{F})$ on A. If $\mathscr{F}'$ is a Nekovář structure that refines $\mathscr{F}$, then $h(\mathscr{F}')$ refines $h(\mathscr{F})$.

(ii) Let $\mathcal{F}$ be a Mazur–Rubin structure on A. Then there exists a Nekovář structure $\mathscr{F}$ on A for which $h(\mathscr{F}) = \mathcal{F}$, $S(\mathscr{F}) = S(\mathcal{F})$ and $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A) = H^1_{\mathcal{F}}(k_\mathfrak{q}, A)[-1]$ for all $\mathfrak{q} \in S(\mathcal{F})$. These conditions determine $\mathscr{F}$ uniquely if and only if $\mathrm{Ext}^1_R(H^1_{\mathcal{F}}(k_\mathfrak{q}, A), H^0(k_\mathfrak{q}, A)) = (0)$ for all $\mathfrak{q}$ in $S(\mathcal{F})$.

Proof. Fix a Nekovář structure $\mathscr{F}$ on A. Then one obtains a well-defined Mazur–Rubin structure $h(\mathscr{F})$ by setting

$$S(h(\mathscr{F})) = S(\mathscr{F}) \quad \text{and} \quad H^1_{h(\mathscr{F})}(k_\mathfrak{q}, A) := \mathrm{im}(H^1(\theta_{\mathscr{F},\mathfrak{q}})) \text{ for all } \mathfrak{q} \in \Pi_k.$$

With this definition, it is also immediately clear that $h(\mathscr{F}') \leq h(\mathscr{F})$ if $\mathscr{F}' \leq \mathscr{F}$, as required to prove (i).

To prove (ii), we use for each $\mathfrak{q}$ in $S(\mathcal{F})$ the convergent cohomological spectral sequence

$$E_2^{p,q} = \prod_{t\in\mathbb{Z}} \mathrm{Ext}^p_R(H^t(H^1_{\mathcal{F}}(k_\mathfrak{q}, A)[-1])), H^{q+t}(\mathrm{R}\Gamma(k_\mathfrak{q}, A)))$$
$$\Rightarrow H^{p+q}(\mathrm{RHom}_R(H^1_{\mathcal{F}}(k_\mathfrak{q}, A)[-1], \mathrm{R}\Gamma(k_\mathfrak{q}, A)))$$

from [109, III, 4.6.10]. Since $H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A)[-1]$ is acyclic outside degree 1, this spectral sequence converges to give a short exact sequence

$$0 \to \mathrm{Ext}^1_R(H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A), H^0(k_{\mathfrak{q}}, A)) \to \mathrm{Hom}_{D(R)}(H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A)[-1], \mathrm{R}\Gamma(k_{\mathfrak{q}}, A)) \xrightarrow{\phi\mapsto H^1(\phi)} \mathrm{Hom}_R(H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A), H^1(k_{\mathfrak{q}}, A)) \to 0.$$

We may therefore choose a morphism $\phi_{\mathfrak{q}}\colon H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A)[-1] \to \mathrm{R}\Gamma(k_{\mathfrak{q}}, A)$ in $D(R)$ for which $H^1(\phi_{\mathfrak{q}})$ is the inclusion $H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A) \to H^1(k_{\mathfrak{q}}, A)$. In particular, if we define $\mathscr{F}$ to be the Nekovář structure with $S(\mathscr{F}) = S(\mathcal{F})$ and, for each $\mathfrak{q} \in S(\mathcal{F})$, both $\mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}}, A) = H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A)[-1]$ and $\theta_{\mathscr{F},\mathfrak{q}} = \theta_{\mathfrak{q}}$, then it is clear $h(\mathscr{F}) = \mathcal{F}$. It is also clear from the displayed exact sequence that, given $\mathcal{F}$, a Nekovář structure with these conditions is unique if and only if $\mathrm{Ext}^1_R(H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A), H^0(k_{\mathfrak{q}}, A))$ vanishes for all $\mathfrak{q}$ in $S(\mathcal{F})$. □

(3.23) Example. Fix a finite subset S of Π_k with $S(A) \subseteq S$.

(i) The relaxed Selmer structure $\mathcal{F}_{\mathrm{rel}} = \mathcal{F}_{\mathrm{rel}}(A, S)$ for A and S defined in [25, Exam. 2.4] is equal to $h(\mathscr{F}_{\mathrm{rel}}(A, S))$. In particular, $S(\mathcal{F}_{\mathrm{rel}}) = S$ and $H^1_{\mathcal{F}_{\mathrm{rel}}}(k_{\mathfrak{q}}, A) = H^1(k_{\mathfrak{q}}, A)$ for $\mathfrak{q} \in S$.

(ii) The canonical Selmer structure $\mathcal{F}_{\mathrm{can}} = \mathcal{F}_{\mathrm{can}}(A, S)$ for A and S defined in [74, Def. 3.2.1] is equal to $h(\mathscr{F}_{\mathrm{can}}(A, S))$. In particular, $S(\mathcal{F}_{\mathrm{can}}) = S$ and

$$H^1_{\mathcal{F}_{\mathrm{can}}}(k_{\mathfrak{q}}, A) = \begin{cases} H^1_f(k_{\mathfrak{q}}, A), & \text{if } \mathfrak{q} \in S \setminus (\Pi_k^\infty \cup \Pi_k^p), \\ H^1(k_{\mathfrak{q}}, A), & \text{if } \mathfrak{q} \in \Pi_k^\infty \cup \Pi_k^p. \end{cases}$$

(iii) If R is a $\mathbb{Z}_p$-order, then the unramified Selmer structure $\mathcal{F}_{\mathrm{ur}} = \mathcal{F}_{\mathrm{ur}}(A, S)$ for A and S is a refinement of $\mathcal{F}_{\mathrm{can}}(A)$ defined in [76, Def. 5.1]. One has $S(\mathcal{F}_{\mathrm{ur}}) = S$ and for $\mathfrak{q} \in S$ sets

$$H^1_{\mathcal{F}_{\mathrm{ur}}}(k_{\mathfrak{q}}, A) \coloneqq \begin{cases} H^1_{\mathcal{F}_{\mathrm{can}}}(k_{\mathfrak{q}}, A), & \text{if } \mathfrak{q} \in S \setminus \Pi_k^p, \\ \big(\bigcap_L \mathrm{Cor}_{L/k_{\mathfrak{q}}}(H^1(L, A))\big)^{\mathrm{sat}}, & \text{if } \mathfrak{q} \in \Pi_k^p, \end{cases}$$

where in the intersection L runs over all finite unramified extensions of $k_{\mathfrak{q}}$, and we write X^{sat} for the saturation of a subgroup X of $H^1(k_{\mathfrak{q}}, A)$. In particular, if R is a discrete valuation ring, then [76, Cor. 5.3] shows that $\mathcal{F}_{\mathrm{ur}}(A)$ coincides with $\mathcal{F}_{\mathrm{can}}(A)$ if and only if $H^0(k_{\mathfrak{q}}, A^\vee(1))$ is finite for every $\mathfrak{q} \in \Pi_k^p$.

(iv) Let A' be a submodule, respectively quotient, of the $R[G_k]$-module A. Then, by [74, Exam. 1.3.3 and 2.17], a Mazur–Rubin structure $\mathcal{F}$ on A induces the Mazur–Rubin structure $\mathcal{F}_{A'}$ on A' with $S(\mathcal{F}_{A'}) \coloneqq S(\mathcal{F})$ and, for each $\mathfrak{q} \in S(\mathcal{F})$,

$$H^1_{\mathcal{F}_{A'}}(k_{\mathfrak{q}}, A') \coloneqq \ker(H^1(k_{\mathfrak{q}}, A') \to H^1_{/\mathcal{F}}(k_{\mathfrak{q}}, A),$$

respectively

$$H^1_{\mathcal{F}_{A'}}(k_{\mathfrak{q}}, A') \coloneqq \mathrm{im}(H^1_{\mathcal{F}}(k_{\mathfrak{q}}, A) \to H^1(k_{\mathfrak{q}}, A')).$$

If $\mathscr{F}$ is a Nekovář structure on A such that $h(\mathscr{F}) = \mathcal{F}$, then, in the respective notation of Example 3.18 (i) and (ii), one has in both cases $\mathcal{F}_{A'} = h(\mathscr{F}_{A'})$.

(3.24) Remark. If A' is a quotient of the $R[G_k]$-module A then, for the following sorts of reasons, care is needed in the use of induced structures in the sense of Example 3.23 (iv).

(i) Whilst, in some cases, the induced structure $\mathcal{F}_{\mathrm{can}}(A)_{A'}$ coincides with $\mathcal{F}_{\mathrm{can}}(A')$, it is in general strictly finer (cf. [92, Lem. 3.5], [74, Prop. 6.2.6]).

(ii) Let $R \to R'$ be a surjective morphism of rings satisfying (2.5), and write A' for the corresponding quotient $A \otimes_R R'$ of A. Assume A is a free R-module, let $\mathscr{F}$ be a Nekovář structure on A and recall the Nekovář structure $\mathscr{F} \otimes_R R'$ on A' defined (under condition (3.1)) in Example 3.18 (iii). Then one has $h(\mathscr{F})_{A'} \le h(\mathscr{F} \otimes_R R')$ and, in general, this refinement is strict. For example, in contrast to the situation for Nekovář structures themselves (cf. Example 3.15 (iv)), the latter refinement is usually strict in the case that $\mathscr{F}$ is a relaxed Nekovář structure.

To each Mazur–Rubin structure one associates a Selmer module as follows.

(3.25) Definition. *Let $\mathcal{F}$ be a Mazur–Rubin structure on A. For $\mathfrak{q} \in \Pi_k$, we write $H^1_{/\mathcal{F}}(k_\mathfrak{q}, A)$ for the quotient module $H^1(k_\mathfrak{q}, A)/H^1_\mathcal{F}(k_\mathfrak{q}, A)$. The Selmer module $H^1_\mathcal{F}(k, A)$ of $\mathcal{F}$ is then defined to be the kernel of the natural localisation map*

$$H^1(\mathcal{O}_{k,S(\mathcal{F})}, A) \xrightarrow{\lambda(\mathcal{F})} \bigoplus\nolimits_{\mathfrak{q}\in S(\mathcal{F})} H^1_{/\mathcal{F}}(k_\mathfrak{q}, A).$$

If S is any finite subset of Π_k with $S(\mathcal{F}) \subseteq S$, then the argument of Lemma 3.12 (i) implies $H^1_\mathcal{F}(k, A)$ is also equal to the kernel of the localisation map

$$\lambda_S(\mathcal{F})\colon H^1(\mathcal{O}_{k,S}, A) \to \bigoplus_{\mathfrak{q}\in S} H^1_{/\mathcal{F}}(k_\mathfrak{q}, A).$$

To describe the link between the Selmer complex of a Nekovář structure $\mathscr{F}$ on A and the Selmer module of the Mazur–Rubin structure $h(\mathscr{F})$, we use, for each finite subset S of Π_k, the natural diagonal map

$$\lambda^0_S(\mathscr{F})\colon H^0(k, A) \to \bigoplus_{\mathfrak{q}\in S} \frac{H^0(k_\mathfrak{q}, A)}{\mathrm{im}(H^0(\theta_{\mathscr{F},\mathfrak{q}}))}. \tag{3.26}$$

(This is an analogue in degree zero of the map $\lambda_S(h(\mathscr{F}))$.)

(3.27) Lemma. *For each Nekovář structure $\mathscr{F}$ on A the following claims are valid.*

(i) There exist canonical short exact sequences of R-modules

$$0 \to \bigoplus\nolimits_{\mathfrak{q}\in S(\mathscr{F})} \ker(H^0(\theta_{\mathscr{F},\mathfrak{q}})) \to H^0_\mathscr{F}(k, A) \to \ker(\lambda^0_{S(\mathscr{F})}(\mathscr{F})) \to 0$$

$$0 \to H^1_{h(\mathscr{F})}(k, A)^\vee \to H^1_\mathscr{F}(k, A)^\vee \to \ker(\lambda^0_{S(\mathscr{F})}(\mathscr{F})^\vee) \to 0.$$

(ii) If S and S' are finite subsets of Π_k with $S' \subseteq S$, then there exists a canonical exact sequence of R-modules

$$0 \to \ker(\lambda^0_{S\setminus S'}(\mathscr{F})^\vee) \to \ker(\lambda^0_S(\mathscr{F})^\vee) \xrightarrow{\alpha} \bigoplus\nolimits_{\mathfrak{q}\in S'} \left(\frac{H^0(k_\mathfrak{q}, A)}{\mathrm{im}(H^0(\theta_{\mathscr{F},\mathfrak{q}}))}\right)^\vee \to \mathrm{cok}(\lambda^0_{S\setminus S'}(\mathscr{F})^\vee) \to 0.$$

Proof. To prove (i) we abbreviate $S(\mathscr{F})$ to S. We then note that $\mathscr{F}$ refines the relaxed structure $\mathscr{F}_{\mathrm{rel}}$ defined in Example 3.15 (i), and hence that the long exact cohomology sequence of the exact triangle of Lemma 3.12 (ii) for the pair $(\mathscr{F}, \mathscr{F}_{\mathrm{rel}})$ gives an exact commutative diagram

$$\begin{array}{ccccccccc}
H^{i-1}(\mathcal{O}_{k,S}, A) & \longrightarrow & \bigoplus H^{i-1}(C_\mathfrak{q}) & \longrightarrow & H^i_\mathscr{F}(k, A) & \to & H^i(\mathcal{O}_{k,S}, A) & \longrightarrow & \bigoplus H^i(C_\mathfrak{q}) \\
\Vert & & \uparrow{\scriptstyle (\beta^{i-1}_\mathfrak{q})_\mathfrak{q}} & & & & \Vert & & \uparrow{\scriptstyle (\beta^i_\mathfrak{q})_\mathfrak{q}} \\
H^{i-1}(\mathcal{O}_{k,S}, A) & \xrightarrow{\lambda^{i-1}} & \bigoplus \dfrac{H^{i-1}(k_\mathfrak{q}, A)}{\mathrm{im}(H^{i-1}(\theta_{\mathscr{F},\mathfrak{q}}))} & & & & H^i(\mathcal{O}_{k,S}, A) & \xrightarrow{\lambda^i} & \bigoplus \dfrac{H^i(k_\mathfrak{q}, A)}{\mathrm{im}(H^i(\theta_{\mathscr{F},\mathfrak{q}}))}
\end{array}$$

Here all direct sums run over $\mathfrak{q} \in S$ and for such $\mathfrak{q}$ we set $C_\mathfrak{q} := \mathrm{R}\Gamma_{\mathscr{F}_{\mathrm{rel}}/\mathscr{F}}(k_\mathfrak{q}, A) = \mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A)$ and write $\beta^i_\mathfrak{q}$ for the injective map induced by the long exact cohomology sequence of the tautological exact triangle $\mathrm{R}\Gamma_\mathscr{F}(k_\mathfrak{q}, A) \to \mathrm{R}\Gamma(k_\mathfrak{q}, A) \to C_\mathfrak{q} \to \cdot$. In addition, λ^i denotes the natural localisation map so that $\lambda^0 = \lambda^0_S(\mathscr{F})$ and $\lambda^1 = \lambda_S(h(\mathscr{F}))$.

Now $H^{-1}(\mathcal{O}_{k,S}, A) = (0)$ and, for each $\mathfrak{q}$, also $H^{-1}(C_\mathfrak{q}) = \ker(H^0(\theta_{\mathscr{F},\mathfrak{q}}))$ since $H^{-1}(k_\mathfrak{q}, A) = (0)$. Given these facts, the exactness of the above diagram with $i = 0$, respectively $i = 1$, directly gives the first exact sequence in (i), respectively a short exact sequence

$$0 \to \mathrm{cok}(\lambda^0_S(\mathscr{F})) \to H^1_\mathscr{F}(k, A) \to H^1_{h(\mathscr{F})}(k, A) \to 0. \tag{3.28}$$

The second exact sequence in (i) is then obtained as the Matlis dual of the latter sequence.

The exact sequence in (ii) is directly obtained by applying the Snake Lemma to the obvious exact commutative diagram

$$\begin{array}{ccccc} \ker(\lambda^0_{S\setminus S'}(\mathscr{F})^\vee) & \hookrightarrow & \left(\bigoplus_{\mathfrak{q}\in S\setminus S'} \frac{H^0(k_\mathfrak{q},A)}{\mathrm{im}(H^0(\theta_{\mathscr{F},\mathfrak{q}}))}\right)^\vee & \xrightarrow{\lambda^0_{S\setminus S'}(\mathscr{F})^\vee} & \mathrm{im}(\lambda^0_{S\setminus S'}(\mathscr{F})^\vee) \\ \downarrow & & \downarrow & & \downarrow \\ \ker(\lambda^0_{S}(\mathscr{F})^\vee) & \hookrightarrow & \left(\bigoplus_{\mathfrak{q}\in S} \frac{H^0(k_\mathfrak{q},A)}{\mathrm{im}(H^0(\theta_{\mathscr{F},\mathfrak{q}}))}\right)^\vee & \xrightarrow{\lambda^0_{S}(\mathscr{F})^\vee} & H^0(k,A)^\vee \\ & & \downarrow & & \\ & & \left(\bigoplus_{\mathfrak{q}\in S'} \frac{H^0(k_\mathfrak{q},A)}{\mathrm{im}(H^0(\theta_{\mathscr{F},\mathfrak{q}}))}\right)^\vee. & & \end{array}$$

□

Let $\mathcal{F}$ be a Mazur–Rubin structure on A and, for each n, write $\mathcal{F}_n$ for the induced structure $\mathcal{F}_{A_n}$ on $A_n = A \otimes_R (R/\mathfrak{a}_n)$. Then the explicit definition of induced structure (in Example 3.23 (iv)) implies the existence of a natural projection map $H^1_{\mathcal{F}_{n+1}}(k, A_{n+1}) \to H^1_{\mathcal{F}_n}(k, A_n)$. In a similar way, there is a natural diagonal projection map from $H^1_{\mathcal{F}}(k, A)$ to the corresponding inverse limit $\varprojlim_{n\in\mathbb{N}} H^1_{\mathcal{F}_n}(k, A_n)$. The following result is the analogue of Proposition 3.19 in this setting (and does not require A to be a free R-module).

(3.29) Lemma. *Assume A is finitely generated as an R-module. Then, for any Mazur–Rubin structure $\mathcal{F}$ on A, the natural map $H^1_{\mathcal{F}}(k, A) \to \varprojlim_{n\in\mathbb{N}} H^1_{\mathcal{F}_n}(k, A_n)$ is bijective.*

Proof. Consider the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathcal{F}}(k,A) & \longrightarrow & H^1(k,A) & \longrightarrow & \bigoplus_{v\in\Pi_k} H^1_{/\mathcal{F}}(k_v,A) \\ & & \downarrow & & \downarrow\simeq & & \downarrow\alpha \\ 0 & \longrightarrow & \varprojlim_n H^1_{\mathcal{F}_n}(k,A_n) & \longrightarrow & \varprojlim_n H^1(k,A_n) & \longrightarrow & \varprojlim_n \bigoplus_{v\in\Pi_k} H^1_{/\mathcal{F}_n}(k_v,A_n). \end{array}$$

Here the middle vertical map is bijective by [92, Prop. B.2.3] since each A_n is finite. It therefore suffices to prove that α is injective and hence, since α factors as a natural composite

$$\bigoplus_v H^1_{/\mathcal{F}}(k_v, A) \xrightarrow{(\beta_v)_v} \bigoplus_v \varprojlim_n H^1_{/\mathcal{F}_n}(k_v, A_n) \hookrightarrow \varprojlim_n \bigoplus_v H^1_{/\mathcal{F}_n}(k_v, A_n),$$

it suffices to show that each map β_v is bijective. To do this, we consider the natural exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathcal{F}}(k_v,A) & \longrightarrow & H^1(k_v,A) & \longrightarrow & H^1_{/\mathcal{F}}(k_v,A) & \longrightarrow & 0 \\ & & \downarrow\gamma_v & & \downarrow\simeq & & \downarrow\beta_v & & \\ 0 & \longrightarrow & \varprojlim_n H^1_{\mathcal{F}_n}(k_v,A_n) & \longrightarrow & \varprojlim_n H^1(k_v,A_n) & \longrightarrow & \varprojlim_n H^1_{/\mathcal{F}_n}(k_v,A_n) & \longrightarrow & 0. \end{array}$$

Here the second vertical map is again bijective because each A_n is finite and the exactness of the bottom row is obtained by applying the functor $\varprojlim_n(-)$ to the underlying (tautological) exact sequence of finite groups. The bijectivity of β_v will therefore follow as a consequence of the Snake Lemma provided that γ_v is surjective. This is in turn true since each map $H^1_{\mathcal{F}}(k_v, A) \to H^1_{\mathcal{F}_n}(k_v, A_n)$ is surjective (by definition of $\mathcal{F}_n$) and inverse limits are exact on the category of finite groups. □

3.3. Duality results

If A is finitely generated over R, then, for each $\mathfrak{q} \in \Pi_k$, local Tate duality induces a canonical isomorphism of R-modules $H^1(k_\mathfrak{q}, A) \simeq H^1(k_\mathfrak{q}, A^\vee(1))^\vee$. Following [74, §1.3], for any Mazur–Rubin structure $\mathcal{F}$ on A, one uses these isomorphisms to define a 'dual' Mazur–Rubin structure

$\mathcal{F}^\vee$ on $A^\vee(1)$ by setting $S(\mathcal{F}^\vee) \coloneqq S(\mathcal{F})$ and, for each $\mathfrak{q} \in \Pi_k$,

$$H^1_{\mathcal{F}^\vee}(k_\mathfrak{q}, A^\vee(1)) \coloneqq \ker(H^1(k_\mathfrak{q}, A^\vee(1)) \simeq H^1(k_\mathfrak{q}, A)^\vee \to H^1_{\mathcal{F}}(k_\mathfrak{q}, A)^\vee) \cong H^1_{/\mathcal{F}}(k_\mathfrak{q}, A)^\vee.$$

In this section, we discuss the natural analogues of this construction for Nekovář structures.

3.3.1. Dual Nekovář–Selmer structures

The following terminology will be convenient.

(3.30) Definition. *A Nekovář structure $\mathscr{F}$ on A is '∞-relaxed', respectively '∞-strict', if for every $\mathfrak{q} \in \Pi_k^\infty$ one has $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A) = \mathrm{R}\Gamma(k_\mathfrak{q}, A)$ and $\theta_{\mathscr{F},\mathfrak{q}}$ is the identity morphism, respectively $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A)$ is the zero complex.*

(3.31) Example. The structures $\mathscr{F}_{\mathrm{rel}}(A,S)_\Sigma$, $\mathscr{F}_{\mathrm{rel}}(A,S)^\Sigma$, $\mathscr{F}_{\mathrm{can}}(A,S)_\Sigma$ and $\mathscr{F}_{\mathrm{can}}(A,S)_\Sigma$ from Example 3.15 are ∞-relaxed, whilst $\mathscr{F}_{\mathrm{str}}(A,S)_\Sigma$ and $\mathscr{F}_{\mathrm{str}}(A,S)^\Sigma$ are ∞-strict.

We now introduce the notions of dual Nekovář structure that are useful in our theory.

(3.32) Definition. *For any Nekovář structure $\mathscr{F}$ on a finite-rank free R-module A, we specify 'dual' ∞-strict Nekovář structures $\mathscr{F}^*$ on $A^*(1)$ and $\mathscr{F}^\vee$ on $A^\vee(1)$ as follows.*

(a) $S(\mathscr{F}^*) = S(\mathscr{F})$; for $\mathfrak{q} \in \Pi_k^\infty$ one has $\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q}, A^*(1)) \coloneqq 0$; for $\mathfrak{q} \in S(\mathscr{F}) \setminus \Pi_k^\infty$, we set

$$\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q}, A^*(1)) \coloneqq \mathrm{RHom}_R(\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A), R[-2])$$

and take $\theta_{\mathscr{F}^*,\mathfrak{q}}$ to be the composite morphism

$$\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q}, A) \xrightarrow{\phi_\mathfrak{q}^*[-2]} \mathrm{RHom}_R(\mathrm{R}\Gamma(k_\mathfrak{q}, A), R[-2]) \cong \mathrm{R}\Gamma(k_\mathfrak{q}, A^*(1))$$

where $\phi_\mathfrak{q}$ is the natural morphism $\mathrm{R}\Gamma(k_\mathfrak{q}, A) \to \mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A)$ and the isomorphism is the first isomorphism in Proposition 3.5 (i).

(b) $S(\mathscr{F}^\vee) = S(\mathscr{F})$; for $\mathfrak{q} \in \Pi_k^\infty$ one has $\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q}, A^\vee(1)) \coloneqq 0$; for $\mathfrak{q} \in S(\mathscr{F}) \setminus \Pi_k^\infty$, we set

$$\mathrm{R}\Gamma_{\mathscr{F}^\vee}(k_\mathfrak{q}, A^\vee(1)) \coloneqq \mathrm{RHom}_R(\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A), E_R(\Bbbk)[-2])$$

and take $\theta_{\mathscr{F}^\vee,\mathfrak{q}}$ to be the composite morphism

$$\mathrm{R}\Gamma_{\mathscr{F}^\vee}(k_\mathfrak{q}, A) \xrightarrow{\phi_\mathfrak{q}^\vee[-2]} \mathrm{RHom}_R(\mathrm{R}\Gamma(k_\mathfrak{q}, A), E_R(\Bbbk)[-2]) \cong \mathrm{R}\Gamma(k_\mathfrak{q}, A^\vee(1))$$

where $\phi_\mathfrak{q}$ is the natural morphism $\mathrm{R}\Gamma(k_\mathfrak{q}, A) \to \mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A)$ and the isomorphism is the second isomorphism in Proposition 3.5 (i).

(3.33) Example. For any finite subset Σ of Π_k with $\Sigma \cap S(\mathscr{F}) = \varnothing$, one has $(\mathscr{F}_{\mathrm{rel}}(A,S)_\Sigma)^* = \mathscr{F}_{\mathrm{str}}(A^*(1),S)^\Sigma$ and $(\mathscr{F}_{\mathrm{rel}}(A,S)_\Sigma)^\vee = \mathscr{F}_{\mathrm{str}}(A^\vee(1),S)^\Sigma$.

The following result establishes some important compatibilities between the notions of dual and induced Nekovář and Mazur–Rubin structures.

(3.34) Lemma. *For every Nekovář structure $\mathscr{F}$ on a finite-rank free R-module A the following claims are valid.*

(i) For a morphism $R \to R'$ of rings satisfying (2.5), the isomorphism $(A \otimes_R R')^*(1) \cong A^*(1) \otimes_R R'$ induces an equality $(\mathscr{F}_{A \otimes_R R'})^* = (\mathscr{F}^*)_{A^*(1) \otimes_R R'}$ of Nekovář structures.

(ii) For every $n \leq m$, the isomorphism $A_n^*(1) \cong A_m^\vee[\mathfrak{a}_n]$ induces an equality $(\mathscr{F}_{A_n})^* = (\mathscr{F}^*_{A_m})_{A_m^\vee[\mathfrak{a}_n](1)}$ of Nekovář structures.

(iii) One has an equality $h(\mathscr{F}^\vee) = h(\mathscr{F})^\vee$ of Mazur–Rubin structures on $A^\vee(1)$.

Proof. The proofs of (i) and (ii) proceed along very similar lines and so we restrict ourselves to the proof of (ii), leaving details for (i) to the attentive reader. For this, we define functors

$$F_1(-) \coloneqq \mathrm{RHom}_{R_n}(- \otimes^{\mathbb{L}}_R R_n, R_n) \quad \text{and} \quad F_2(-) \coloneqq \mathrm{RHom}_R(R_n, \mathrm{RHom}_R(-, E_R(\Bbbk))).$$

Then it follows from derived Tensor-Hom adjunction [111, Th. 10.8.7] that there is natural isomorphism of functors $F_1 \cong F_2$, hence for every $\mathfrak{q} \in \Pi_k$ we have a commutative diagram

$$\begin{array}{ccc} F_1(\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A_m)) & \xrightarrow{F_1(\theta_{\mathscr{F},\mathfrak{q}})} & F_1(\mathrm{R}\Gamma(k_\mathfrak{q}, A_m)) \\ \downarrow \simeq & & \downarrow \simeq \\ F_2(\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A_m)) & \xrightarrow{F_1(\theta_{\mathscr{F},\mathfrak{q}})} & F_2(\mathrm{R}\Gamma(k_\mathfrak{q}, A_m)) \end{array}$$

In addition, we have an isomorphism (using Lemma 3.3 (vi) for the first and Theorem 3.5 (i) for the second isomorphism)

$$t_1 \colon F_1(\mathrm{R}\Gamma(k_\mathfrak{q}, A_m)) \xrightarrow{\simeq} \mathrm{RHom}_{R_n}(\mathrm{R}\Gamma(k_\mathfrak{q}, A_n), R_n) \xrightarrow{\simeq} \mathrm{R}\Gamma(k_\mathfrak{q}, A_n^\vee(1))$$

and also an isomorphism (using Theorem 3.5 (i) for the first and Lemma 3.3 (vii) for the second isomorphism)

$$t_2 \colon F_2(\mathrm{R}\Gamma(k_\mathfrak{q}, A_m)) \xrightarrow{\simeq} \mathrm{RHom}_R(R_n, \mathrm{R}\Gamma(k_\mathfrak{q}, A_m^\vee(1))) \xrightarrow{\simeq} \mathrm{R}\Gamma(k_\mathfrak{q}, A_n^\vee(1)).$$

By definition, one then has $\theta_{(\mathscr{F}_{A_n})^*,\mathfrak{q}} = t_1 \circ F_1(\theta_{\mathscr{F},\mathfrak{q}})$ and $\theta_{(\mathscr{F}^\vee)_{A_m^\vee[\mathfrak{a}_n](1)},\mathfrak{q}} = t_2 \circ F_2(\theta_{\mathscr{F},\mathfrak{q}})$.
Recall that the isomorphisms of local Tate duality in Theorem 3.5 (i) are induced by cup products and are therefore functorial. It follows that we have a commutative diagram (cf. [80, proof of Prop. 5.2.4])

$$\begin{array}{ccc} F_1(\mathrm{R}\Gamma(k_\mathfrak{q}, A_m)) & \xrightarrow{t_1} & \mathrm{R}\Gamma(k_\mathfrak{q}, A_n^\vee(1)) \\ \downarrow \simeq & & \| \\ F_2(\mathrm{R}\Gamma(k_\mathfrak{q}, A_m)) & \xrightarrow{t_2} & \mathrm{R}\Gamma(k_\mathfrak{q}, A_n^\vee(1)), \end{array}$$

which together with the previous discussion proves the claimed equality $(\mathscr{F}_{A_n})^* = (\mathscr{F}^\vee)_{A_m^\vee[\mathfrak{a}_n](1)}$ of Nekovář structures.
To prove (iii), we recall that, for $\mathfrak{q} \in \Pi_k$, the group $H^1_{h(\mathscr{F})}(k_\mathfrak{q}, A^\vee(1))$ is defined to be the image of $H^1(\theta_{\mathscr{F}^\vee,\mathfrak{q}})$. Now, there is the commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathscr{F}^\vee}(k_\mathfrak{q}, A^\vee(1)) & = & \mathrm{RHom}_R(\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A^*(1)), E_R(\Bbbk))[-2] \\ \downarrow \theta_{\mathscr{F}^*,\mathfrak{q}} & & \downarrow \theta^\vee_{\mathscr{F},\mathfrak{q}} \\ \mathrm{R}\Gamma(k_q, A^\vee(1)) & \xleftarrow{\simeq} & \mathrm{RHom}_R(\mathrm{R}\Gamma(k_q, A), E_R(\Bbbk))[-2]. \end{array}$$

Here the equality in the top line is by definition of $\mathscr{F}^\vee$, and the bottom isomorphism is the second in Proposition 3.5 (i). Taking cohomology then induces, since $E_R(\Bbbk)$ is self-injective and therefore the exact functor $(-)^\vee = \mathrm{Hom}_R(-, E_R(\Bbbk))$ commutes with cohomology, a commutative diagram

$$\begin{array}{ccc} H^1_{\mathscr{F}^\vee}(k_\mathfrak{q}, A^\vee(1)) & = & H^1_{/\mathscr{F}^\vee}(k_\mathfrak{q}, A^\vee(1))^\vee \\ \downarrow H^1(\theta_{\mathscr{F},\mathfrak{q}}) & & \downarrow H^1(\theta_{\mathscr{F},\mathfrak{q}})^\vee \\ H^1(k_\mathfrak{q}, A^\vee(1)) & \xleftarrow{\simeq} & H^1(k_\mathfrak{q}, A)^\vee, \end{array}$$

where bottom isomorphism is by local Tate duality. We deduce that

$$\begin{aligned} H^1_{h(\mathscr{F})}(k_\mathfrak{q}, A^\vee(1)) &\coloneqq \mathrm{im}(H^1(\theta_{\mathscr{F}^\vee,\mathfrak{q}})) \\ &= \mathrm{im}\left(H^1_{/\mathscr{F}}(k_\mathfrak{q}, A^\vee(1))^* \xrightarrow{H^1(\theta_{\mathscr{F},\mathfrak{q}})^\vee} H^1(k_\mathfrak{q}, A)^\vee \cong H^1(k_\mathfrak{q}, A^\vee(1))\right) \\ &\cong \ker\left(\mathrm{im}(\theta_{\mathscr{F},\mathfrak{q}})^\vee \to H^1(k_\mathfrak{q}, A)^\vee \cong H^1(k_\mathfrak{q}, A^\vee(1))\right) \\ &= \ker\left(H^1_{h(\mathscr{F})}(k_\mathfrak{q}, A)^\vee \to H^1(k_\mathfrak{q}, A)^\vee \cong H^1(k_\mathfrak{q}, A^\vee(1))\right) \\ &\eqqcolon H^1_{h(\mathscr{F})^\vee}(k_\mathfrak{q}, A^\vee(1)), \end{aligned}$$

where the isomorphism is obtained by applying the exact functor $(-)^\vee$ to the exact sequence

$$H^1_{\mathscr{F}}(k_\mathfrak{q}, A) \xrightarrow{H^1(\theta_{\mathscr{F},\mathfrak{q}})} H^1(k_\mathfrak{q}, A) \to H^1_{/\mathscr{F}}(k_\mathfrak{q}, A).$$

This proves the claimed equality $h(\mathscr{F}^\vee) = h(\mathscr{F})^\vee$ of Mazur–Rubin structures. □

(3.35) Remark. Assume A (and hence also $A^*(1)$) is a finite-rank free R-module and $\mathscr{F}^*$ is perfect. Then, by combining the identifications $(\mathscr{F}^*)_{A_n^*(1)} = (\mathscr{F}_{A_n})^*$ in Lemma 3.34 (i) with the result of Proposition 3.19 (iii) for $\mathscr{F}^*$, one obtains a natural isomorphism $\mathrm{R}\Gamma_{\mathscr{F}^*}(k, A^*(1)) \cong \varprojlim_n \mathrm{R}\Gamma_{(\mathscr{F}_{A_n})^*}(k, A_n^*(1))$ in $D(R)$ and hence also, in each degree i, an induced identification $H^i_{\mathscr{F}^*}(k, A^*(1)) = \varprojlim_{n\in\mathbb{N}} H^i_{(\mathscr{F}_{A_n})^*}(k, A_n^*(1))$.

(3.36) Remark. For any Mazur–Rubin structure $\mathcal{F}$ on a finitely generated R-module A there exists for each n an equality of Mazur–Rubin structures $(\mathcal{F}_{A_n})^* = (\mathcal{F}^\vee)_{A^\vee[\mathfrak{a}_n](1)}$ on $A_n^*(1) \cong A^\vee[\mathfrak{a}_n]$ that is analogous to Lemma 3.34 (ii) (cf. [74, Ex. 1.3.3]). This combines with the identification $A_n^* \cong A_n^\vee = A_{n+1}^\vee[\mathfrak{a}_n] \hookrightarrow A_{n+1}^\vee = A_{n+1}^*$ (cf. Lemma 2.8) to induce both a morphism $H^1_{(\mathcal{F}_{A_n})^*}(k, A_n^*(1)) \to H^1_{(\mathcal{F}_{A_{n+1}})^*}(k, A_{n+1}^*(1))$ and also a diagonal map $H^1_{\mathcal{F}^\vee}(k, A^\vee(1)) \to \varinjlim_{n\in\mathbb{N}} H^1_{(\mathcal{F}_{A_n})^*}(k, A_n^*(1))$, where the limit is taken with respect to the above morphisms. By a similar argument to that in Lemma 3.29 one can prove that this diagonal map is bijective.

We will also use the following technical result regarding the $\mathfrak{a}_n$-torsion submodules of dual Selmer modules.

(3.37) Lemma. *Assume that A is R-free of finite rank for which $\bar{A}^\vee(1)^{G_k} = (0)$. Then, for any Mazur–Rubin structure $\mathcal{F}$ on A, the natural map $A_n^\vee(1) \hookrightarrow A^\vee(1)$ induces an isomorphism $H^1_{(\mathcal{F}_{A_n})^\vee}(k, A_n^\vee(1)) \xrightarrow{\simeq} H^1_{\mathcal{F}^\vee}(k, A^\vee(1))[\mathfrak{a}_n]$ for every $n \geq 0$.*

Proof. This is proved in [26, Cor. 3.8] (see also [74, Lem. 3.5.3]). □

3.3.2. Consequences of Artin–Verdier duality

In this section, we discuss relations between the Selmer complexes that are respectively associated to a Nekovář structure $\mathscr{F}$ and to its duals $\mathscr{F}^*$ and $\mathscr{F}^\vee$.
For each $\mathfrak{q} \in \Pi_k^\mathbb{R}$ we write $\tau_\mathfrak{q}(A)$ for the canonical morphism $\mathrm{R}\Gamma(k_\mathfrak{q}, A) \to \mathrm{R}\Gamma_{\mathrm{Tate}}(k_\mathfrak{q}, A)$, and we then define an object of $D(R)$ by setting

$$\Delta_\infty(k, A) := \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\mathbb{C}} A \oplus \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\mathbb{R}} \mathrm{Cone}(\tau_\mathfrak{q}(A))[-1].$$

From Proposition 3.5 (ii) one then obtains the following relation between the Selmer complex for $\mathscr{F}$ and the appropriate duals of the Selmer complexes for $\mathscr{F}^*$ and $\mathscr{F}^\vee$.

(3.38) Proposition. *Artin–Verdier duality induces canonical morphisms*

$$\mu_{\mathscr{F},\mathscr{F}^*}\colon \mathrm{RHom}_R(\Delta_\infty(k, A^*(1)), R)[-3] \to \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\infty} \mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A)$$
$$\mu_{\mathscr{F},\mathscr{F}^\vee}\colon \mathrm{RHom}_R(\Delta_\infty(k, A^\vee(1)), E_R(\Bbbk))[-3] \to \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\infty} \mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A)$$

in $D(R)$. Setting $\mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^*}(k, A) := \mathrm{Cone}(\mu_{\mathscr{F},\mathscr{F}^*})$ and $\mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^\vee}(k, A) := \mathrm{Cone}(\mu_{\mathscr{F},\mathscr{F}^\vee})$, there exist canonical exact triangles in $D(R)$

$$\mathrm{R}\Gamma_{\mathscr{F}}(k, A) \xrightarrow{\delta_{\mathscr{F},\mathscr{F}^*}} \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathscr{F}^*}(k, A^*(1)), R)[-3] \to \mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^*}(k, A) \to \cdot$$
$$\mathrm{R}\Gamma_{\mathscr{F}}(k, A) \xrightarrow{\delta_{\mathscr{F},\mathscr{F}^\vee}} \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathscr{F}^\vee}(k, A^\vee(1)), E_R(\Bbbk))[-3] \to \mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^\vee}(k, A) \to \cdot.$$

Proof. To prove the stated results for $\mathscr{F}^*$, we set $B := A^*(1)$ and consider the following commutative diagram in $D(R)$

$$\begin{tikzcd} & & \Delta_\infty(k,B) \arrow[d] & \\ \mathrm{R}\Gamma_c(\mathcal{O}_{k,S},B) \arrow[r] \arrow[d, "\mu"] & \mathrm{R}\Gamma(\mathcal{O}_{k,S},B) \arrow[r] \arrow[d, equal] & \bigoplus_{\mathfrak{q}\in S}\mathrm{R}\Gamma(k_\mathfrak{q},B) \arrow[r] \arrow[d, "(\tau_\mathfrak{q}(B))_\mathfrak{q}"] & \cdot \\ \widetilde{\mathrm{R}\Gamma}_c(\mathcal{O}_{k,S},B) \arrow[r] & \mathrm{R}\Gamma(\mathcal{O}_{k,S},B) \arrow[r, "\lambda_S(B)"] & \bigoplus_{\mathfrak{q}\in S}\mathrm{R}\Gamma_{\mathrm{Tate}}(k_\mathfrak{q},B) \arrow[r] \arrow[d] & \cdot \\ & & \cdot & \end{tikzcd} \tag{3.39}$$

Here the upper triangle is the appropriate case of the exact triangle (3.2) and, for $\mathfrak{q} \in S \setminus \Pi_k^{\mathbb{R}}$, we write $\mathrm{R}\Gamma_{\mathrm{Tate}}(k_\mathfrak{q}, B)$ and $\tau_\mathfrak{q}(A)$ for $\mathrm{R}\Gamma(k_\mathfrak{q}, B)$ and the identity morphism $\mathrm{R}\Gamma(k_\mathfrak{q}, B) \to \mathrm{R}\Gamma(k_\mathfrak{q}, B)$. In addition, $\lambda_S(B)$ is the natural localisation morphism and $\widetilde{\mathrm{R}\Gamma}_c(\mathcal{O}_{k,S}, B)$ denotes its mapping fibre and the vertical exact triangle is that which follows directly from the definition of $\Delta_\infty(k, B)$. In particular, from the (obvious) commutativity of the central square, one deduces the existence of a morphism μ that completes the diagram to give a morphism of exact triangles, and thereby implies the existence of an exact triangle that forms the first column of the following commutative diagram in $D(R)$

$$\begin{tikzcd} \mathrm{R}\Gamma_c(\mathcal{O}_{k,S},B) \arrow[r] \arrow[d, "\mu"] & \mathrm{R}\Gamma_{\mathscr{F}^*}(k,B) \arrow[r] \arrow[d, "\mu_{\mathscr{F}^*}"] & \bigoplus_{\mathfrak{q}\in S\setminus\Pi_k^\infty}\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q},B) \arrow[r] \arrow[d, equal] & \cdot \\ \widetilde{\mathrm{R}\Gamma}_c(\mathcal{O}_{k,S},B) \arrow[r] \arrow[d] & \widetilde{\mathrm{R}\Gamma}_{\mathscr{F}^*}(k,B) \arrow[r] \arrow[d, "\mu'_{\mathscr{F}^*}"] & \bigoplus_{\mathfrak{q}\in S\setminus\Pi_k^\infty}\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q},B) \arrow[r] & \cdot \\ \Delta_\infty(k,B) \arrow[r, equal] \arrow[d] & \Delta_\infty(k,B) \arrow[d] & & \\ \cdot & \cdot & & \end{tikzcd} \tag{3.40}$$

Here the complex $\widetilde{\mathrm{R}\Gamma}_{\mathscr{F}^*}(k, B)$ is defined via the same exact triangle as $\mathrm{R}\Gamma_{\mathscr{F}^*}(k, B)$ after replacing $\mathrm{R}\Gamma(k_\mathfrak{q}, B)$ for each $\mathfrak{q} \in \Pi_k^\infty$ by $\mathrm{R}\Gamma_{\mathrm{Tate}}(k_\mathfrak{q}, B)$ and the morphism $\mu_{\mathscr{F}^*}$ is then induced by the obvious analogue of (3.39). In addition, the first row is the exact triangle of (3.14) (with A and $\mathscr{F}$ replaced by B and $\mathscr{F}^*$) and the second row is the natural analogue of this exact triangle. In particular, since the first square in the diagram commutes there exists a morphism $\mu_{\mathscr{F}^*}$ which makes the lower square commute and the second column an exact triangle in $D(R)$. Now, if we apply the exact functor $X \mapsto X^*[-3]$ to the second row of (3.40) and then substitute the first isomorphism in Proposition 3.5 (ii) and recall the explicit definition of the complexes $\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q}, B)$ for $\mathfrak{q} \in S \setminus \Pi_k^\infty$, we obtain an exact triangle in $D(R)$

$$\widetilde{\mathrm{R}\Gamma}_{\mathscr{F}^*}(k, B)^*[-3] \to \mathrm{R}\Gamma(\mathcal{O}_{k,S}, A) \xrightarrow{\lambda} \bigoplus_{\mathfrak{q}\in S\setminus\Pi_k^\infty} \mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q}, A) \to \cdot$$

in which λ is the natural localisation map. By the argument of Lemma 3.12 (ii), this in turn induces a canonical exact triangle in $D(R)$ that forms the central row of the following diagram

$$
\begin{array}{ccccccc}
 & & \Delta_\infty(k,B)^*[-3] & \xrightarrow{\theta'\circ\mu'} & \bigoplus_{\mathfrak{q}\in\Pi_k^\infty}\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q},A) & & \\
 & & \Big\downarrow{\scriptstyle \mu'} & & \Big\| & & \\
\mathrm{R}\Gamma_{\mathscr{F}}(k,A) & \xrightarrow{\theta} & \widetilde{\mathrm{R}\Gamma}_{\mathscr{F}^*}(k,B)^*[-3] & \xrightarrow{\theta'} & \bigoplus_{\mathfrak{q}\in\Pi_k^\infty}\mathrm{R}\Gamma_{/\mathscr{F}}(k_\mathfrak{q},A) & \longrightarrow & \cdot \\
\Big\downarrow{\scriptstyle \mu\circ\theta} & & \Big\downarrow{\scriptstyle \mu} & & & & \\
\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q},B)^*[-3] & = & \mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q},B)^*[-3] & & & & \\
 & & \Big\downarrow & & & & \\
 & & \cdot & & & &
\end{array}
$$

Here the central column is the exact triangle obtained by applying the functor $X \mapsto X^*[-3]$ to the second column in (3.40) so $\mu' = (\mu'_{\mathscr{F}^*})^*[-3]$ and $\mu = (\mu_{\mathscr{F}^*})^*[-3]$. In particular, if we respectively define $\mu_{\mathscr{F},\mathscr{F}^*}$ and $\delta_{\mathscr{F},\mathscr{F}^*}$ to be the morphisms $\theta' \circ (\mu'_{\mathscr{F}^*})^*[-3]$ and $(\mu_{\mathscr{F}^*})^*[-3] \circ \theta$, then the Octahedral axiom combines with the commutativity of the above diagram to imply that the mapping fibres of $\mu_{\mathscr{F},\mathscr{F}^*}$ and $\delta_{\mathscr{F},\mathscr{F}^*}$ are isomorphic (in $D(R)$). This last fact leads directly to an exact triangle of the required form and so proves all claimed results for $\mathscr{F}^*$.
In addition, after making obvious changes, the same argument derives the analogous claims for $\mathscr{F}^\vee$ from the second isomorphism in Proposition 3.5 (ii). Since this is a routine matter, we leave details to an interested reader. □

(3.41) Remark. If $\mathscr{F}$ is ∞-relaxed, then $\mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^\vee}(k,A) = \mathrm{RHom}_R(\Delta_\infty(k,A^\vee(1)),E_R(\Bbbk))[-2]$ and, by explicit computation, one checks that this is represented by the complex

$$\Big(\bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^{\mathbb{C}}}A(-1)[-2]\Big)\oplus\Big(\bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^{\mathbb{R}}}[A(-1)\xrightarrow{1-c_\mathfrak{q}}A(-1)\xrightarrow{1+c_\mathfrak{q}}A(-1)\xrightarrow{1-c_\mathfrak{q}}A(-1)\xrightarrow{1+c_\mathfrak{q}}\cdots]\Big).$$

Here, for each $\mathfrak{q}\in\Pi_k^{\mathbb{R}}$, we write $c_\mathfrak{q}$ for the non-trivial element of $G_{k_\mathfrak{q}}$ and the first term of the displayed complex occurs in degree 2. If $\mathscr{F}$ is ∞-relaxed and either R is self-injective or A is a free R-module, then $\mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^*}(k,A)$ also coincides with the above complex.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Mazur–Rubin structures on A with $\mathcal{F}_1 \le \mathcal{F}_2$. Then Mazur and Rubin [74, Th. 2.3.4] have shown that global duality gives rise to a canonical exact sequence of R-modules

$$H^1_{\mathcal{F}_1}(k,A)\hookrightarrow H^1_{\mathcal{F}_2}(k,A)\to\bigoplus_{\mathfrak{q}\in S(\mathcal{F}_1)}\frac{H^1_{\mathcal{F}_2}(k_\mathfrak{q},A)}{H^1_{\mathcal{F}_1}(k_\mathfrak{q},A)}\to H^1_{\mathcal{F}_1^*}(k,A^\vee(1))^\vee\twoheadrightarrow H^1_{\mathcal{F}_2^*}(k,A^\vee(1))^\vee. \quad (3.42)$$

We next use Proposition 3.38 to establish an analogue of this result for Nekovář structures.

(3.43) Theorem. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be ∞-relaxed Nekovář structures on A with $\mathscr{F}_1 \le \mathscr{F}_2$ and set $B := A^\vee(1)$. For $i\in\{1,2\}$ write $\tilde{H}^1_{\mathscr{F}_i^\vee}(k_\mathfrak{q},B)^\vee$ for the kernel of the morphism*

$$H^1_{\mathscr{F}_i^\vee}(k_\mathfrak{q},B)^\vee\to H^2\big(\mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F}_i}(k,A)\big)$$

induced by the exact triangle in Proposition 3.38. Then the following claims are valid.

(i) There exists a canonical long exact sequence of R-modules

$$\bigoplus H^0\big(\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_\mathfrak{q},A)\big)\to H^1_{\mathscr{F}_1}(k,A)\to H^1_{\mathscr{F}_2}(k,A)\to\bigoplus H^1\big(\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_\mathfrak{q},A)\big)$$
$$\to\tilde{H}^1_{\mathscr{F}_1^\vee}(k_\mathfrak{q},B)^\vee\to\tilde{H}^1_{\mathscr{F}_2^\vee}(k,B)^\vee\to\bigoplus H^0\big(\mathrm{R}\Gamma_{\mathscr{F}_1^\vee/\mathscr{F}_2^\vee}(k_\mathfrak{q},B)\big)^\vee$$

in which all direct sums are taken over $\mathfrak{q}\in S(\mathscr{F}_1)$.

(ii) Set $\mathcal{F}_1 := h(\mathscr{F}_1)$ and $\mathcal{F}_2 := h(\mathscr{F}_2)$ and assume the following hypotheses:

(a) the maps $H^1(\theta_{\mathscr{F}_1,\mathfrak{q}})$, $H^1(\theta_{\mathscr{F}_2,\mathfrak{q}})$ and $H^2(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ are injective for every $\mathfrak{q}\in S(\mathscr{F}_1)$:

(b) $\Pi_k^{\mathbb{R}}=\varnothing$ if $p=2$.

Then there exists a canonical exact commutative diagram of R-modules

$$\begin{array}{ccccccccc}
H^1_{\mathscr{F}_1}(k,A) & \xrightarrow{\alpha_1} & H^1_{\mathscr{F}_2}(k,A) & \longrightarrow & \bigoplus H^1\big(\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_{\mathfrak{q}},A)\big) & \longrightarrow & \tilde{H}^1_{\mathscr{F}_1^\vee}(k_{\mathfrak{q}},B)^\vee & \xrightarrow{\alpha_2} & \tilde{H}^1_{\mathscr{F}_2^\vee}(k,B)^\vee \\
\Big\downarrow{\scriptstyle \beta_1} & & \Big\downarrow{\scriptstyle \beta_2} & & {\scriptstyle \beta_3}\Big\| & & {\scriptstyle \beta_4}\Big\uparrow & & {\scriptstyle \beta_5}\Big\uparrow \\
H^1_{\mathcal{F}_1}(k,A) & \hookrightarrow & H^1_{\mathcal{F}_2}(k,A) & \longrightarrow & \bigoplus \dfrac{H^1_{\mathcal{F}_2}(k_{\mathfrak{q}},A)}{H^1_{\mathcal{F}_1}(k_{\mathfrak{q}},A)} & \longrightarrow & H^1_{\mathcal{F}_1^\vee}(k_{\mathfrak{q}},B)^\vee & \twoheadrightarrow & H^1_{\mathcal{F}_2^\vee}(k,B)^\vee.
\end{array}$$

Here both direct sums are taken over $\mathfrak{q} \in S(\mathscr{F}_1)\cup S(\mathscr{F}_2) = S(\mathcal{F}_1)\cup S(\mathcal{F}_2)$, the first row is induced by the exact sequence in (i), the second row is the relevant case of (3.42) and all vertical maps are described in the course of the argument below. In addition, the map α_1 is injective (resp. α_2 is surjective) if for each $\mathfrak{q} \in S(\mathscr{F}_1)\cup S(\mathscr{F}_2)$ the map $H^0(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ is surjective (resp. $H^2(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ is surjective and $H^3(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ is injective).

Proof. We set $S \coloneqq S(\mathscr{F}_1)\cup S(\mathscr{F}_2)$. We then recall that, since $\mathscr{F}_1 \le \mathscr{F}_2$, for each $\mathfrak{q}\in S$ we are given a morphism $j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}}\colon \mathrm{R}\Gamma_{\mathscr{F}_1}(k_{\mathfrak{q}},A)\to \mathrm{R}\Gamma_{\mathscr{F}_2}(k_{\mathfrak{q}},A)$ in $D(R)$ and $\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_{\mathfrak{q}},A)$ denotes its mapping cone.
Now, since $\mathscr{F}_1$ and $\mathscr{F}_2$ are both ∞-relaxed, the morphism $j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}}$ is an isomorphism for each $\mathfrak{q}\in \Pi_k^\infty$. In addition, if we set $B \coloneqq A^\vee(1)$, then for each $\mathfrak{q}\in S\setminus \Pi_k^\infty$, the definition of the dual condition $(\mathrm{R}\Gamma_{\mathscr{F}^\vee}(k_{\mathfrak{q}},B),\theta_{\mathscr{F}_i^\vee,\mathfrak{q}})$ implies that $j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}}$ induces a canonical morphism

$$j_{\mathscr{F}_2^\vee,\mathscr{F}_1^\vee,\mathfrak{q}}\colon \mathrm{R}\Gamma_{\mathscr{F}_2^\vee}(k_{\mathfrak{q}},B)\to \mathrm{R}\Gamma_{\mathscr{F}_1^\vee}(k_{\mathfrak{q}},B)$$

whose mapping cone is isomorphic to $\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_{\mathfrak{q}},A)^\vee[-2]$. In particular, if we use these morphisms to regard $\mathscr{F}_2^\vee$ as a refinement of $\mathscr{F}_1^\vee$, then we obtain a commutative diagram of exact triangles in $D(R)$ of the form

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{\mathscr{F}_1}(k,A) & \longrightarrow & \mathrm{R}\Gamma_{\mathscr{F}_1^\vee}(k,A^\vee(1))^\vee[-3] & \longrightarrow & \mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F}_1,\mathscr{F}_1^\vee}(k,A) \longrightarrow \cdot \\
\Big\downarrow & & \Big\downarrow & & \Big\| \\
\mathrm{R}\Gamma_{\mathscr{F}_2}(k,A) & \longrightarrow & \mathrm{R}\Gamma_{\mathscr{F}_2^\vee}(k,A^\vee(1))^\vee[-3] & \longrightarrow & \mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F}_2,\mathscr{F}_2^\vee}(k,A) \longrightarrow \cdot \\
\Big\downarrow & & \Big\downarrow & & \\
\bigoplus_{\mathfrak{q}\in S}\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_{\mathfrak{q}},A) & \underset{\delta}{\xrightarrow{\cong}} & \bigoplus_{\mathfrak{q}\in S}\mathrm{R}\Gamma_{\mathscr{F}_1^\vee/\mathscr{F}_2^\vee}(k_{\mathfrak{q}},A)^\vee[-2] & & \\
\Big\downarrow & & \Big\downarrow & & \\
\cdot & & \cdot & &
\end{array}$$

Here the first and second rows are the respective exact triangles from Proposition 3.38 and the equality occurs since $\mathscr{F}_1$ and $\mathscr{F}_2$ are ∞-relaxed (cf. Remark 3.41). In addition, the first column is the exact triangle in Lemma 3.12 (ii) with $(\mathscr{F}',\mathscr{F}) = (\mathscr{F}_1,\mathscr{F}_2)$ and the second is the image under the exact functor $X\mapsto X^*[-3]$ of the corresponding exact triangle with $(\mathscr{F}',\mathscr{F}) = (\mathscr{F}_2^\vee,\mathscr{F}_1^\vee)$. Finally, δ is the direct sum over $\mathfrak{q}$ of the local duality isomorphisms described above. The exact sequence in (i) is now obtained by combining the long exact cohomology sequences of the first two columns.
In regard to (ii), we first note that the assumed injectivity, for $\mathfrak{q}\in S$, of $H^1(\theta_{\mathscr{F}_1,\mathfrak{q}})$ and $H^1(\theta_{\mathscr{F}_2,\mathfrak{q}})$ implies that $H^1(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ is injective, and then combines with the assumed injectivity of $H^2(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ to induce an identification $H^1\big(\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_{\mathfrak{q}},A)\big)\cong H^1_{\mathcal{F}_2}(k_{\mathfrak{q}},A)/H^1_{\mathcal{F}_1}(k_{\mathfrak{q}},A)$. We take the map β_3 in the claimed diagram to be the direct sum over $\mathfrak{q}\in S$ of these identifications. We next define β_1 and β_2 to be the Matlis duals of the first map in the second exact sequence of Lemma 3.27 (i) with $\mathscr{F}$ taken to be $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. We also note that, for both $i=1$ and $i=2$, the same exact sequence in Lemma 3.27 (i) with $\mathscr{F}$ taken to be $\mathscr{F}_i^\vee$ induces

an exact sequence

$$0 \to H^1_{\mathcal{F}_i^\vee}(k,B)^\vee \xrightarrow{\gamma_i} H^1_{\mathscr{F}_i^\vee}(k,B)^\vee \to \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\infty} H^0(k_\mathfrak{q},B)^\vee \oplus \bigoplus\nolimits_{\mathfrak{q}\in S(\mathscr{F}_i)\setminus\Pi_k^\infty} \left(\frac{H^0(k_\mathfrak{q},B)}{\mathrm{im}(H^0(\theta_{\mathscr{F}_i^\vee,\mathfrak{q}}))}\right)^\vee$$

in which in the direct sum we have also used the fact that $\mathscr{F}_i^\vee$ is ∞-strict so that $H^0(\theta_{\mathscr{F}_i^\vee,\mathfrak{q}})$ is the zero map for each $\mathfrak{q}\in\Pi_k^\infty$. In addition, under condition (c), there are natural isomorphisms

$$\begin{aligned}\bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\infty} H^0(k_\mathfrak{q},B)^\vee &\cong \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\mathbb{C}} B^\vee \oplus \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\mathbb{R}} \big((1+c_\mathfrak{q})B\big)^\vee\\ &\cong \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\mathbb{C}} A \oplus \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\mathbb{R}} (1-c_\mathfrak{q})A\\ &= H^2\big(\mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F}}(k,A)\big),\end{aligned}$$

where the equality follows from Remark 3.41. It follows that the map γ_i factors through the submodule $\tilde{H}^1_{\mathscr{F}_1^\vee}(k_\mathfrak{q},B)^\vee$ of $H^1_{\mathscr{F}_1^\vee}(k_\mathfrak{q},B)^\vee$ and so, in the claimed diagram, we can take β_4 and β_5 to be the maps that are respectively induced by γ_1 and γ_2. With these specifications of the maps β_i, it is then a straightforward exercise to check that the claimed diagram commutes, as required.

Finally, we note that the assumed surjectivity of $H^0(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ and injectivity of $H^1(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ for all $\mathfrak{q}\in S$ would combine to imply $H^0\big(\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_\mathfrak{q},A)\big)=(0)$ for such $\mathfrak{q}$ and hence that α_1 is injective as a consequence of the exact sequence in (i). In a similar way, the surjectivity of $H^2(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ and injectivity of $H^3(j_{\mathscr{F}_1,\mathscr{F}_2,\mathfrak{q}})$ for all $\mathfrak{q}\in S$ would imply $H^2\big(\mathrm{R}\Gamma_{\mathscr{F}_2/\mathscr{F}_1}(k_\mathfrak{q},A)\big)=(0)$ for such $\mathfrak{q}$ and hence that α_2 is surjective. This verifies the final assertion of (ii). □

3.4. Perfect Selmer complexes

In this section we assume $\Pi_k^\mathbb{R}=\varnothing$ if $p=2$. We also fix a Nekovář structure $\mathscr{F}$ on a finitely generated free R-module A and assume the following hypothesis to be valid.

(3.44) Hypothesis. The following conditions are satisfied:

- (a) $\mathscr{F}$ is ∞-relaxed;
- (b) For every $\mathfrak{q}\in S(\mathscr{F})\setminus\Pi_k^\infty$, the following conditions are satisfied:
  - (i) $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q},A)$ belongs to $D^{\mathrm{perf}}_{[0,2]}(R)$;
  - (ii) $H^0(\theta_{\mathscr{F},\mathfrak{q}})$ is injective;
- (c) For some $\mathfrak{q}_0\in S(\mathscr{F})\setminus\Pi_k^\infty$, the map $H^0(\theta_{\mathscr{F},\mathfrak{q}_0})$ is 0.
- (d) For some $\mathfrak{q}_1\in S(\mathscr{F})\setminus\Pi_k^\infty$, the map $H^2(\theta_{\mathscr{F},\mathfrak{q}_1})$ is bijective.

For each finite subset S of Π_k we also use the map $\lambda^0_S(\mathscr{F}^\vee)$ from (3.26) to define an R-module

$$X_S(\mathscr{F}) := \ker(\lambda^0_S(\mathscr{F}^\vee)^\vee).$$

We then abbreviate $X_{S(\mathscr{F})}(\mathscr{F})$ to $X(\mathscr{F})$.

The following result constructs a family of complexes that plays a key role in our theory.

(3.45) Proposition. *Assume $\Pi_k^\mathbb{R}=\varnothing$ if $p=2$, that A is a free R-module of finite rank, and that $\mathscr{F}$ satisfies Hypothesis 3.44. Then the following claims are valid.*

(i) The complex

$$C(\mathscr{F}) := \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathscr{F}^*}(k,A^*(1)),R)[-2]$$

is a well-defined object of $D^{\mathrm{perf}}(R)$ such that, in $K_0(R)$, one has

$$\chi_R\big(C(\mathscr{F})\big) = \sum\nolimits_{\mathfrak{q}\in S(\mathscr{F})\setminus\Pi_k^\infty} \chi_R\big(\mathrm{R}\Gamma_{\mathscr{F}^*}(k_\mathfrak{q},A^*(1))\big).$$

(ii) $C(\mathscr{F})$ is canonically isomorphic to $\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathscr{F}^\vee}(k,A^\vee(1)),E_R(\Bbbk))[-2]$ *in $D(R)$. In particular, in each degree i, one has $H^i(C(\mathscr{F}))=H^{2-i}_{\mathscr{F}^\vee}(k,A^\vee(1))^\vee$.*

(iii) *Fix a morphism $R \to R'$ of rings satisfying (2.5). Then there exists a natural isomorphism $C(\mathscr{F}) \otimes^{\mathbb{L}}_R R' \cong C(\mathscr{F} \otimes_R R')$ in $D^{\mathrm{perf}}(R')$, where $\mathscr{F} \otimes_R R'$ is the Nekovář structure on $A \otimes_R R'$ defined in Example 3.18 (iii).*

(iv) *Lemma 2.29 defines an object $\varprojlim_{n\in\mathbb{N}} C(\mathscr{F}_{A_n})$ of $D^{\mathrm{perf}}(R)$ that is naturally isomorphic to $C(\mathscr{F})$. In particular, in each degree i, one has $H^i(C(\mathscr{F})) = \varprojlim_{n\in\mathbb{N}} H^{2-i}_{(\mathscr{F}_{A_n})^*}(k, A_n^*(1))^*$.*

(v) *$C(\mathscr{F})$ is acyclic outside degrees 0 and 1. There exists a canonical identification $H^0(C(\mathscr{F})) = H^1_{\mathscr{F}}(k, A)$ and a canonical exact commutative diagram*

$$\begin{array}{ccccc}
X_{S(\mathscr{F})\setminus\Pi_k^\infty}(\mathscr{F}) & & & & \\
{\scriptstyle\alpha_1}\big\uparrow & & & & \\
H^2_{\mathscr{F}}(k,A) & \hookrightarrow & H^1(C(\mathscr{F})) & \twoheadrightarrow & \bigoplus_{\mathfrak{q}\in\Pi_k^\infty} H^0(k_{\mathfrak{q}}, A^\vee(1))^\vee \\
{\scriptstyle\alpha_2}\big\uparrow & & \big\| & & {\scriptstyle\alpha_3}\big\uparrow \\
H^1_{h(\mathscr{F})^\vee}(k, A^\vee(1))^\vee & \hookrightarrow & H^1(C(\mathscr{F})) & \twoheadrightarrow & X(\mathscr{F}).
\end{array} \tag{3.46}$$

(vi) *Let Σ be a finite subset of Π_K. Then, for the Σ-comodification $\mathscr{F}^\Sigma$ of $\mathscr{F}$ defined in Example 3.16, there exists a canonical exact triangle in $D^{\mathrm{perf}}(R)$*

$$C(\mathscr{F}) \to C(\mathscr{F}^\Sigma) \xrightarrow{(\rho_{\mathfrak{q}})_{\mathfrak{q}}} \bigoplus_{\mathfrak{q}\in\Sigma\setminus S(\mathscr{F})} \mathrm{R}\Gamma_f(k_{\mathfrak{q}}, A^*(1))^*[-1] \to \cdot \tag{3.47}$$

Proof. We set $B \coloneqq A^*(1)$. Since $\mathscr{F}^*$ is ∞-strict and $S(\mathscr{F}^*) = S(\mathscr{F})$, Lemma 3.12 (iii) implies $\mathscr{F}^*$ is perfect if and only if, for every $\mathfrak{q} \in S(\mathscr{F}) \setminus \Pi_k^\infty$, the complex $\mathrm{R}\Gamma_{\mathscr{F}^*}(k_{\mathfrak{q}}, B)$ belongs to $D^{\mathrm{perf}}(R)$. Since the latter condition is satisfied as a consequence of Hypothesis 3.44 (b)(i), it follows that $\mathrm{R}\Gamma_{\mathscr{F}^*}(k, B)$, and hence also $C(\mathscr{F})$, belongs to $D^{\mathrm{perf}}(R)$. For similar reasons, one also has

$$\chi_R(C(\mathscr{F})) = \chi_R(\mathrm{R}\Gamma_{\mathscr{F}^*}(k_{\mathfrak{q}}, B)) = \sum\nolimits_{\mathfrak{q}\in S(\mathscr{F})\setminus\Pi_k^\infty} \chi_R(\mathrm{R}\Gamma_{\mathscr{F}^*}(k_{\mathfrak{q}}, B),$$

where the second equality follows from the formula in Lemma 3.12 (iii) with $(A, \mathscr{F})$ replaced by $(B, \mathscr{F}^*)$. This proves (i).

To prove (ii), we use the following diagram in $D(R)$

$$\begin{array}{ccccccc}
C(\mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^*}(k,A)[1] & \xrightarrow{\theta_1} & \mathrm{R}\Gamma_{\mathscr{F}}(k,A)[2] & \longrightarrow & \cdot \\
\big\downarrow & & \| & & \| & & \\
\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathscr{F}^\vee}(k, A^\vee(1)), E_R(\Bbbk))[-2] & \longrightarrow & \mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F},\mathscr{F}^\vee}(k,A)[1] & \xrightarrow{\theta_2} & \mathrm{R}\Gamma_{\mathscr{F}}(k,A)[2] & \longrightarrow & \cdot
\end{array}$$

Here the upper and lower rows are equivalent to the respective exact triangles in Proposition 3.38 (ii) and the left hand equality follows from Remark 3.41. In addition, an analysis of the construction of these triangles shows that the morphisms θ_1 and θ_2 coincide as they are both induced by the morphism $\bigoplus_{\mathfrak{q}\in\Pi_k^\infty} \mathrm{R}\Gamma_{/\mathscr{F}}(k_{\mathfrak{q}}, A) \to \mathrm{R}\Gamma_{\mathscr{F}}(k, A)[1]$ that occurs in the exact triangle (3.10). The Octahedral axiom therefore implies the existence of a dashed arrow that makes the above diagram into a morphisms of exact triangles and hence is an isomorphism in $D(R)$. This proves the first assertion of (ii) and then the second assertion follows immediately from the fact that Matlis duality is an exact functor.

To prove (iii) we note that the argument of Lemma 3.34 (i) implies an equality $\mathscr{F}^* \otimes_R R' = (\mathscr{F} \otimes_R R')^*$ of Nekovář structures on $A^*(1) \otimes_R R' = (A \otimes_R R')^*(1)$. After taking this into account, the exact triangle (3.14) with $\mathscr{F}$ replaced by $\mathscr{F}^*$ combines with Lemma 3.3 (vi) (with $\mathscr{F}(-)$ taken to be $\mathrm{R}\Gamma_c(\mathcal{O}_{k,S}, -)$) and the explicit definition of the local conditions for $\mathscr{F}^* \otimes_R R'$ to imply the existence of a natural isomorphism in $D(R')$

$$\mathrm{R}\Gamma_{\mathscr{F}^*}(k, A^*(1)) \otimes^{\mathbb{L}}_R R' \cong \mathrm{R}\Gamma_{(\mathscr{F}\otimes_R R')^*}(k, (A \otimes_R R')^*(1)).$$

Upon applying the exact functor $\mathrm{RHom}_{R'}(-, R')[-2]$ to this isomorphism one obtains the claimed isomorphism in (ii).

To prove (iv) we first apply (iii) with $R \to R'$ taken to be $R \to R/\mathfrak{a}_n$ to deduce that for each n there exists a natural isomorphism $C(\mathscr{F})\otimes^{\mathbb{L}}_R R/\mathfrak{a}_n \cong C(\mathscr{F}_{A_n})$ in $D^{\mathrm{perf}}(R/\mathfrak{a}_n)$. Given this family of isomorphisms the first assertion of (iii) is proved by mimicking the argument of Proposition 3.19 after replacing $\mathrm{R}\Gamma_{\mathscr{F}}(k,A)$ by $C(\mathscr{F})$. From the ismorphism $C(\mathscr{F}) \cong \varprojlim_{n\in\mathbb{N}} C(\mathscr{F}_{A_n})$ one then derives, in each degree i, an identification

$$H^i(C(\mathscr{F})) = H^i(\varprojlim_{n\in\mathbb{N}} C(\mathscr{F}_{A_n})) = \varprojlim_{n\in\mathbb{N}} H^i(C(\mathscr{F}_{A_n})) = \varprojlim_{n\in\mathbb{N}} H^{2-i}_{(\mathscr{F}_{A_n})^*}(k, A_n^*(1))^*.$$

Here the second equality is valid since inverse limits are exact on the category of finite abelian groups and the third since R_n is self-injective and so taking duals commutes with taking cohomology. This proves (iv).

Turning to the proof of (v), we note that, since $\Pi_k^{\mathbb{R}} = \varnothing$ if $p = 2$, Hypothesis 3.44 (a) implies $H^i_{\mathscr{F}}(k_{\mathfrak{q}}, A) = (0)$ for all $i > 2$ and all $\mathfrak{q} \in \Pi_k^\infty$. This fact combines with Remark 3.41 to imply $H^i(\mathrm{R}\Gamma^{\mathrm{AV}}_{\mathscr{F}}(k,A)) = (0)$ for $i \neq 2$ and also combines with 3.44 (b) (iii) and (d) and Lemma 3.12 (iv) to imply $H^i_{\mathscr{F}}(k,A) = (0)$ for $i < 0$ and $i > 2$. These observations in turn combine with the long cohomology exact sequence of the exact triangle in Proposition 3.38 to imply $H^i(C(\mathscr{F})) = (0)$ for $i < -1$ and $i > 1$, to identify $H^i(C(\mathscr{F}))$ with $H^{i+1}_{\mathscr{F}}(k,A)$ for $i \in \{-1,0\}$ and to give a short exact sequence that forms the central row of (3.46). In addition, since 3.44 (c) implies injectivity of $\lambda^0_{S(\mathscr{F})}(\mathscr{F})$, 3.44 (b) (ii) combines with the first exact sequence in Lemma 3.27 (ii) to imply that $H^0_{\mathscr{F}}(k,A) = (0)$. To complete the proof of (ii), it is therefore enough to construct the diagram (3.46). To do this, we note that $H^1(C_{\mathscr{F}}(A)) \cong H^1_{\mathscr{F}^\vee}(k, A^\vee(1))^\vee$ by (iv) and the fact that taking Matlis duals is exact. Given this identification, we obtain the exact sequence that forms the lower row of (3.46) by first taking Matlis duals in the second exact sequence in Lemma 3.27 (i) with $(A,\mathscr{F})$ taken to be $(A^\vee(1),\mathscr{F}^\vee)$, and then recalling that $h(\mathscr{F}^\vee) = h(\mathscr{F})^\vee$ by Lemma 3.34 (iii).

We now take the map α_3 in (3.46) to be the map α in the exact sequence of Lemma 3.27 (iii) with $(\mathscr{F}, S)$ replaced by $(\mathscr{F}^\vee, S(\mathscr{F}^\vee))$. Then, with this definition, the commutativity of the second square in (3.46) is clear and this has two consequences: firstly, the map α_3 is surjective (as can also be seen directly from Lemma 3.27 (iii) since the conditions 3.44 (b)(iii) and (d) combine to imply $H^0(\theta_{\mathscr{F}^\vee,\mathfrak{q}_1})$ is the zero map) and there exists an injective morphism α_2 that makes the first square of (3.46) commute. The existence of a surjective morphism α_1 that makes the first column of (3.46) a short exact sequence now follows by applying the Snake Lemma to the lower two rows of the diagram and taking account of the exact sequence in Lemma 3.27 (ii) with $(\mathscr{F}, S, S')$ taken to be $(\mathscr{F}^\vee, S(\mathscr{F}^\vee), \Pi_k^\infty)$. This proves (v).

To prove (vi) we note that $(\mathscr{F}^\Sigma)^*$ coincides with the Nekovář structure $(\mathscr{F}^*)_\Sigma$ on B. Given this, the claimed exact triangle is directly obtained by applying the exact functor $X \mapsto X^*[-2]$ to the exact triangle (3.17) with $(A,\mathscr{F})$ replaced by $(B,\mathscr{F}^*)$. □

The following result establishes properties of the modules $X_S(\mathscr{F})$ used in later arguments.

(3.48) Lemma. *Let $\mathscr{F}$ be a Nekovář structure on a finitely generated R-module A and S a finite subset of Π_k. Then the R-module $X_S(\mathscr{F}) := \ker(\lambda^0_S(\mathscr{F}^\vee)^\vee)$ is finitely generated. Further, if condition (3.1) is valid, A is a free R-module and $\mathrm{R}\Gamma_{\mathscr{F}}(k_{\mathfrak{q}}, A)$ belongs to $D^{\mathrm{perf}}_{[0,2]}(R)$ for all $\mathfrak{q} \in S$, then the following claims are also valid.*

- *(i) Every morphism $R \to R'$ of rings satisfying condition (2.5) induces a surjective map of R'-modules $X_S(\mathscr{F}) \otimes_R R' \twoheadrightarrow X_S(\mathscr{F} \otimes_R R')$.*
- *(ii) The maps from (i) combine to give an isomorphism $X_S(\mathscr{F}) \cong \varprojlim_{i\in\mathbb{N}} X_S(\mathscr{F} \otimes_R R_i)$ of R-modules in which all of the transition morphisms in the inverse limit are surjective.*

Proof. The definition of $\lambda^0_S(\mathscr{F}^\vee)$ directly implies that

$$X_S(\mathscr{F}) \subseteq \bigoplus_{\mathfrak{q}\in S}\big(H^0(k_{\mathfrak{q}}, A^\vee(1))/\operatorname{im}(H^0(\theta_{\mathscr{F}^\vee,\mathfrak{q}}))\big)^\vee \subseteq \bigoplus_{\mathfrak{q}\in S} H^0(k_{\mathfrak{q}}, A^\vee(1))^\vee.$$

In addition, for each $\mathfrak{q} \in S$, the Matlis dual of the inclusion $H^0(k_{\mathfrak{q}}, A^\vee(1)) \subseteq A^\vee(1)$ is a surjective map $A(-1) \twoheadrightarrow H^0(k_v, A^\vee(1))^\vee$. The latter map implies that each R-module $H^0(k_{\mathfrak{q}}, A^\vee(1))^\vee$

is finitely generated and hence, since S is finite (and R is Noetherian), the displayed inclusions imply $X_S(\mathscr{F})$ is also finitely generated over R, as claimed.
In the rest of the argument we assume all of the stated hypotheses for (i) and (ii). We then fix $\mathfrak{q} \in S$ and note that the definition of $\mathscr{F}^\vee$ gives an idomorphism

$$\begin{aligned}\ker\big(H^0(k_\mathfrak{q}, A^\vee(1))^\vee \xrightarrow{H^0(\theta_{\mathscr{F}^\vee,\mathfrak{q}})^\vee} H^0_{\mathscr{F}^\vee}(k_\mathfrak{q}, A^\vee(1))^\vee\big) &\cong \ker\big(H^2(k_\mathfrak{q}, A) \to H^2_{/\mathscr{F}}(k_\mathfrak{q}, A)\big)\\ &= \operatorname{im}\big(H^2_{\mathscr{F}}(k_\mathfrak{q}, A) \xrightarrow{H^2(\theta_{\mathscr{F}})} H^2(k_\mathfrak{q}, A)\big).\end{aligned}$$

Consider now the commutative diagram

$$\begin{array}{ccccccc} H^1_{/\mathscr{F}}(k_\mathfrak{q}, A) \otimes_R R' & \longrightarrow & H^2_{\mathscr{F}}(k_\mathfrak{q}, A) \otimes_R R' & \xrightarrow{H^2(\theta_{\mathscr{F}})} & (\operatorname{coker} H^0(\theta_{\mathscr{F}^\vee}))^\vee \otimes_R R' & \longrightarrow & 0 \\ \downarrow & & \downarrow{\cong} & & \downarrow & & \\ H^1_{/\mathscr{F}}(k_\mathfrak{q}, A \otimes_R R') & \longrightarrow & H^2_{\mathscr{F}}(k_\mathfrak{q}, A \otimes_R R') & \xrightarrow{H^2(\theta_{\mathscr{F}\otimes_R R'})} & (\operatorname{coker} H^0(\theta_{(\mathscr{F}\otimes_R R')^\vee}))^\vee & \longrightarrow & 0, \end{array}$$

where the middle arrow is an isomorphism by the assumption that $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A) \in D^{\mathrm{perf}}_{[0,2]}(R)$ and (the argument of) Lemma 2.31 (ii). It follows that the rightmost arrow is surjective. In addition, global duality (Proposition 3.5 (ii)) shows that $H^0(k, A^\vee(1))^\vee \cong H^3(\widetilde{\mathrm{R}\Gamma}_c(k, A))$ and since $\widetilde{\mathrm{R}\Gamma}_c(k, A)$ belongs to $D^{\mathrm{perf}}_{[0,3]}(R)$ (by condition (3.1)), and so we similarly deduce that the map $H^0(k, A^\vee(1))^\vee \otimes_R R' \to H^0(k, (A \otimes_R R')^\vee(1))^\vee$ is an isomorphism.
From the definition of $X_S(\mathscr{F})$ (resp. of $X_S(\mathscr{F} \otimes_R R')$) we then obtain the commutative diagram

$$\begin{array}{ccccccccc} & & X_S(\mathscr{F}) \otimes_R R' & \longrightarrow & \big(\bigoplus_{\mathfrak{q}\in S}(\operatorname{coker} H^0(\theta_{\mathscr{F}^\vee}))^\vee\big) \otimes_R R' & \longrightarrow & H^0(k, A^\vee(1))^\vee \otimes_R R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_S(\mathscr{F} \otimes_R R') & \longrightarrow & \bigoplus_{\mathfrak{q}\in S}(\operatorname{coker} H^0(\theta_{(\mathscr{F}\otimes_R R')^\vee}))^\vee & \longrightarrow & H^0(k, (A \otimes_R R')^\vee(1))^\vee & \longrightarrow & 0 \end{array}$$

We have seen above that the two rightmost arrows are isomorphisms, hence the arrow on the left is as well. This proves claim (i).
As for claim (ii), the above diagram reduces us to proving, for every $\mathfrak{q} \in S$, the isomorphisms

$$\varprojlim_{i\in\mathbb{N}}(\operatorname{coker} H^0(\theta_{(\mathscr{F}\otimes_R R_i)^\vee}))^\vee \cong (\operatorname{coker} H^0(\theta_{\mathscr{F}^\vee}))^\vee \qquad \varprojlim_{i\in\mathbb{N}} H^0(k, (A\otimes_R R_i)^\vee(1))^\vee \cong H^0(k, A^\vee(1))^\vee.$$

As in the discussion of claim (i), we can then further reduce to establishing the isomorphisms $\varprojlim_{i\in\mathbb{N}} H^2_{\mathscr{F}\otimes_R R_i}(k_\mathfrak{q}, A \otimes_R R_i) \cong H^2_{\mathscr{F}}(k_\mathfrak{q}, A)$ and $\varprojlim_{i\in\mathbb{N}} H^2(k_\mathfrak{q}, A \otimes_R R_i) \cong H^2_{\mathscr{F}}(k_q, A)$, as well as $\varprojlim_{i\in\mathbb{N}} H^3(\widetilde{\mathrm{R}\Gamma}_c(k, A \otimes_R R_i)) \cong H^3(\widetilde{\mathrm{R}\Gamma}_c(k, A))$. These latter isomorphisms are in turn a consequence of Lemma 2.29, applied to the complexes $\mathrm{R}\Gamma_{\mathscr{F}}(k_\mathfrak{q}, A)$, $\mathrm{R}\Gamma(k_\mathfrak{q}, A)$, and $\widetilde{\mathrm{R}\Gamma}_c(k, A)$. □

4. Euler systems relative to Nekovář structures

4.1. The definition of higher-rank Euler systems

Fix a prime number p and assume condition (3.1). We also fix a ring $\mathcal{R}$ that satisfies condition (2.5), a number field k, and finite-rank free $\mathcal{R}$-module $\mathcal{T}$ that carries a continuous action of G_k. Assume that $S_{\mathrm{ram}}(\mathcal{T})$ is finite, and choose a finite subset S_0 of Π_k with

$$\Pi_k^\infty \cup \Pi_k^p \subseteq S_0.$$

(In applications one often takes $S_0 = \Pi_k^\infty \cup \Pi_k^p \cup S_{\mathrm{ram}}(\mathcal{T})$.) For $\mathfrak{q} \in \Pi_k \setminus S_0$, we set

$$\mathrm{Eul}_\mathfrak{q}(X) := \det(1 - \mathrm{Frob}_\mathfrak{q}^{-1} X \mid H^0(I_\mathfrak{q}, \mathcal{T}^*(1))) \in \mathcal{R}[X],$$

where $I_\mathfrak{q}$ is a choice of inertia subgroup (in G_k) at $\mathfrak{q}$.
We fix an abelian pro-p extension $\mathcal{K}$ of k in which all places in Π_k^∞ split completely, and denote

the collection of finite extensions of k contained in $\mathcal{K}$ by Ω. For each extension K of k contained in $\mathcal{K}$ we abbreviate $S_{\rm ram}(K/k)$ to $S_{\rm ram}(K)$ and then set

$$S_0(K) := S_0 \cup S_{\rm ram}(K) \quad \text{ and } \quad \mathcal{G}_K := \mathrm{Gal}(K/k).$$

We note that $\mathcal{R}[\mathcal{G}_K]$ satisfies condition (2.5) (with the Gorenstein property following, for example, from Lemma 2.3) and, for any subgroup H of $\mathcal{G}_K$, we set

$$N_H := \sum_{\sigma\in H} \sigma \in \mathcal{R}[\mathcal{G}_K].$$

We also write $\mathcal{T}_K$ for the $\mathcal{R}[\mathcal{G}_K] \times \mathcal{R}[\![G_k]\!]$-module $\mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}[\mathcal{G}_K]$ upon which $\mathcal{G}_K$ acts via right multiplication and $\sigma \in G_k$ by $\sigma \cdot (a \otimes x) := (\sigma a) \otimes (x\overline{\sigma}^{-1})$ where $\overline{\sigma}$ is the image of σ in $\mathcal{G}_K$.
In the sequel we assume to be given a family of Nekovář structures on the modules $\mathcal{T}_K$ that satisfies the following hypothesis.

(4.1) Hypothesis. $\mathfrak{F} := (\mathscr{F}_K)_{K\in\Omega}$ is a family of Nekovář structures parametrised by fields in Ω that has both of the following properties.

(i) Each $\mathscr{F}_K$ is a Nekovář structure on $\mathcal{T}_K$ that satisfies Hypothesis 3.44.

(ii) For K and L in Ω with $K \subseteq L$, the Nekovář structure $\mathscr{F}_L \otimes_{\mathcal{R}[\mathcal{G}_L]} \mathcal{R}[\mathcal{G}_K]$ induced by $\mathscr{F}_L$ on $\mathcal{T}_K$ is the modification $\mathscr{F}_K^L := \mathscr{F}_K^{S_{\rm ram}(L)}$ of $\mathscr{F}_K$ defined in Example 3.16.

For K in Ω, we set $H^1_{\mathfrak{F}}(K, \mathcal{T}) := H^1_{\mathscr{F}_K}(k, \mathcal{T}_K)$.

(4.2) Example. Fix a finite subset S' of $\Pi_k \setminus \Pi_k^\infty$ with $\Pi_k^p \cup S_{\rm ram}(\mathcal{T}) \subseteq S'$ and assume to be given, for every $\mathfrak{q} \in S'$, an $\mathcal{R}[G_{k_\mathfrak{q}}]$-submodule $\mathcal{T}_\mathfrak{q}$ of $\mathcal{T}$ that is free as an $\mathcal{R}$-module. As in Example 3.15 we define a Greenberg–Nekovář structure $\mathscr{F}_K := \mathscr{F}((\mathcal{T}_{\mathfrak{q},K}, j_{\mathfrak{q},K})_{\mathfrak{q}\in S(K)})$ by taking

- $S(K) := S' \cup \Pi_k^\infty \cup S_{\rm ram}(K)$,
- $\mathcal{T}_{\mathfrak{q},K} := \mathcal{T}_\mathfrak{q} \otimes_{\mathcal{R}} \mathcal{R}[\mathcal{G}_K]$ if $\mathfrak{q} \in S'$ and $\mathcal{T}_{\mathfrak{q},K} := \mathcal{T}_K$ if $\mathfrak{q} \in S(K) \setminus S'$,
- $j_{\mathfrak{q},K}$ to be the inclusion $\mathcal{T}_{\mathfrak{q},K} \hookrightarrow \mathcal{T}_K$ for all $\mathfrak{q} \in S(K)$.

The family $\mathfrak{F} := (\mathscr{F}_K)_{K\in\Omega}$ then satisfies Hypothesis 3.44 if there exist places $\mathfrak{q}_0, \mathfrak{q}_1 \in S'$ with $\mathcal{T}_{\mathfrak{q}_0} = (0)$ and $\mathcal{T}_{\mathfrak{q}_1} = \mathcal{T}$.

For each K and L in Ω with $K \subseteq L$, the long exact cohomology sequence of the exact triangle (3.47) (with $\mathscr{F}$ taken to be $\mathscr{F}_K$ and Σ to be $S_{\rm ram}(L)$) combines with the explicit descriptions of cohomology given in Proposition 3.45(v) to give an injective map of $\mathcal{R}[\mathcal{G}_K]$-modules

$$H^1_{\mathfrak{F}}(K, \mathcal{T}) = H^1_{\mathscr{F}_K}(k, \mathcal{T}_K) \to H^1_{\mathscr{F}_K^L}(k, \mathcal{T}_K). \tag{4.3}$$

These maps have two important consequences. Firstly, since Hypothesis 4.1 (ii) combines with Proposition 3.45 (iii) and Lemma 2.31 (i) to identify $H^1_{\mathscr{F}_K^L}(k, \mathcal{T})$ with a submodule of $H^1_{\mathfrak{F}}(L, \mathcal{T})$, the map (4.3) induces a canonical injective homomorphism

$$H^1_{\mathfrak{F}}(K, \mathcal{T}) \hookrightarrow H^1_{\mathfrak{F}}(L, \mathcal{T}). \tag{4.4}$$

We use this homomorphism to identify $H^1_{\mathfrak{F}}(K, \mathcal{T})$ with a submodule of $H^1_{\mathfrak{F}}(L, \mathcal{T})$.
In addition, Lemma 2.17 (i) implies that, for each $a \in \mathbb{N}_0$, the map (4.3) induces a canonical injective map of $\mathcal{R}[\mathcal{G}_K]$-modules

$$\bigcap\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathfrak{F}}(K, \mathcal{T}) \hookrightarrow \bigcap\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}_K^L}(k, \mathcal{T}_K).$$

We use these maps to identify $\bigcap^a_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathfrak{F}}(K, \mathcal{T})$ as a submodule of $\bigcap^r_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}_K^L}(k, \mathcal{T}_K)$. We will then also use the isomorphisms constructed in the next result.

(4.5) Lemma. *Fix a family $\mathfrak{F}$ of Nekovář structures as in Hypothesis 4.1 and fields K and L in Ω with $K \subseteq L$. Then, for each natural number a, there exists a natural isomorphism of $\mathcal{R}[\mathcal{G}_K]$-modules*

$$\nu^a_{L/K} \colon \bigcap\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}_K^L}(k, \mathcal{T}_K) \xrightarrow{\simeq} \big(\bigcap\nolimits^a_{\mathcal{R}[\mathcal{G}_L]} H^1_{\mathfrak{F}}(L, \mathcal{T})\big)^{\mathrm{Gal}(L/K)}.$$

Proof. Hypothesis 4.1 (i) implies that the complex $C(\mathscr{F}_L)$ constructed in Proposition 3.45 satisfies the assumptions of Lemma 2.35. It follows that $C(\mathscr{F}_L)$ has a resolution $P^\bullet$ in $D(\mathcal{R}[\mathcal{G}_L])$ of the form $P^0 \xrightarrow{\phi} P^1$, in which P^0 and P^1 are finitely generated free $\mathcal{R}[\mathcal{G}_L]$-modules and P^0 is placed in degree 0. Set $H \coloneqq \mathrm{Gal}(L/K)$ and $P^i_H \coloneqq P^i \otimes_{\mathcal{R}[\mathcal{G}_L]} \mathcal{R}[\mathcal{G}_K]$ for $i \in \{0,1\}$. Then, by applying Lemma 2.17 (i) and taking $\mathrm{Gal}(L/K)$-invariants, we obtain the upper row in the following exact commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \big(\bigcap^a_{\mathcal{R}[\mathcal{G}_L]} H^1_{\mathfrak{F}}(L,\mathcal{T})\big)^H & \longrightarrow & \big(\bigwedge^a_{\mathcal{R}[\mathcal{G}_L]} P^0\big)^H & \xrightarrow{\phi} & \big(P^1 \otimes_{\mathcal{R}[\mathcal{G}_L]} \bigwedge^{a-1}_{\mathcal{R}[\mathcal{G}_L]} P^0\big)^H \\
& \simeq \big\uparrow & & \simeq \big\uparrow & & \simeq \big\uparrow \\
0 \longrightarrow & \bigcap^a_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}^L_K}(k,\mathcal{T}_K) & \longrightarrow & \bigwedge^a_{\mathcal{R}[\mathcal{G}_K]} P^0_H & \xrightarrow{\phi} & P^1_H \otimes_{\mathcal{R}[\mathcal{G}_K]} \bigwedge^{a-1}_{\mathcal{R}[\mathcal{G}_K]} P^0_H.
\end{array}$$

To describe the lower row we note Proposition 3.45 (iii) and Hypothesis 4.1 (ii) combine to imply $C(\mathscr{F}^L_K) = C(\mathscr{F}_L \otimes_{\mathcal{R}[\mathcal{G}_L]} \mathcal{R}[\mathcal{G}_K])$ is isomorphic to $C(\mathscr{F}_L) \otimes^{\mathbb{L}}_{\mathcal{R}[\mathcal{G}_L]} \mathcal{R}[\mathcal{G}_K]$ and hence to $P^\bullet \otimes_{\mathcal{R}[\mathcal{G}_L]} \mathcal{R}[\mathcal{G}_K]$. The lower row is therefore obtained by applying Lemma 2.17 (i) to the exact sequence obtained from the latter resolution. Now, for any finitely generated free $\mathcal{R}[\mathcal{G}_L]$-module M, the assignment $m \mapsto \mathrm{N}_H(m)$ induces an isomorphism $M_H \xrightarrow{\simeq} M^H$. The first and second solid vertical isomorphisms are then obtained by applying this observation with M taken to be $\bigwedge^a_{\mathcal{R}[\mathcal{G}_L]} P^0$ and $P^1 \otimes_{\mathcal{R}[\mathcal{G}_L]} \bigwedge^{a-1}_{\mathcal{R}[\mathcal{G}_L]} P^0$ respectively. Since the square involving these isomorphisms clearly commutes, there exists an induced dashed map as in the diagram and the Snake Lemma implies this map is bijective. We can therefore take the latter map to be the required isomorphism $\nu^a_{L/K}$. □

We can now specify an appropriate notion of Euler system for our theory.

(4.6) Definition. *Let $\mathfrak{F}$ be a family of Nekovář structures as in Hypothesis 4.1. Then, for each natural number a, an 'Euler system' of rank a for $\mathfrak{F}$ is an element*

$$c = (c_K)_{K\in\Omega} \in \prod\nolimits_{K\in\Omega} \bigcap\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathfrak{F}}(K,\mathcal{T})$$

with the property that, for all fields K and L in Ω with $K \subseteq L$, one has

$$\mathrm{N}_{\mathrm{Gal}(L/K)}(c_L) = \nu^a_{L/K}\big(\big(\prod\nolimits_{v\in S_0(L)\setminus S_0(K)} \mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\big) \cdot c_K\big).$$

The collection of all such elements is naturally a module over $\mathcal{R}[\![\mathcal{G}_\mathcal{K}]\!]$ that we denote by $\mathrm{ES}^a_{S_0}(\mathfrak{F})$. If $S_0 = \Pi^\infty_k \cup \Pi^p_k \cup S_{\mathrm{ram}}(\mathcal{T})$, then we write $\mathrm{ES}^a(\mathfrak{F})$ in place of $\mathrm{ES}^a_{S_0}(\mathfrak{F})$.

(4.7) Remark. Assume $\mathcal{R}$ is reduced and $\mathbb{Z}$-torsion free and write $\mathcal{Q}$ for its total field of fractions $\mathcal{Q}$. Then $\mathcal{R}[\mathcal{G}_K]$ is reduced, with total field of fractions $\mathcal{Q}[\mathcal{G}_K]$ and, for every natural number a and finitely generated $\mathcal{R}[\mathcal{G}_K]$-module M, there exists a natural isomorphism

$$\big\{x \in \mathcal{Q} \otimes_\mathcal{R} \bigwedge\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} M \mid f(x) \in \mathcal{R}[\mathcal{G}_K] \text{ for all } f \in \bigwedge\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} M^*\big\} \xrightarrow{\simeq} \bigcap\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} M$$

by Lemma 2.28. In particular, there is also an identification

$$\bigwedge\nolimits^a_{\mathcal{Q}[\mathcal{G}_K]} (\mathcal{Q} \otimes_\mathcal{R} M) \xrightarrow{\simeq} \mathcal{Q} \otimes_\mathcal{R} \bigcap\nolimits^a_{\mathcal{R}[\mathcal{G}_K]} M. \tag{4.8}$$

(4.9) Example. Assume $\mathcal{R}$ is as in Remark 4.7 and, for each K in Ω, write $\mathscr{F}_{\mathrm{rel},K}$ for the relaxed Nekovář structure on $\mathcal{T}_K$. Let $\mathfrak{F}_{\mathrm{rel}} = \mathfrak{F}_{\mathrm{rel}}(\mathcal{T})$ denote the family $\{\mathscr{F}_{\mathrm{rel},K}\}_{K\in\Omega}$ and, for each K in Ω, set $U_K \coloneqq \mathcal{O}_{K,S(K)}$. Then for $L \in \Omega$ with $K \subseteq L$, one has

$$H^1_{\mathfrak{F}_{\mathrm{rel}}}(K,\mathcal{T}) = H^1(U_K,\mathcal{T}) \qquad H^1_{(\mathscr{F}_{\mathrm{rel},K})^L}(k,\mathcal{T}_K) = H^1(\mathcal{O}_{K,S(L)},\mathcal{T}) \qquad H^1_{\mathfrak{F}_{\mathrm{rel}}}(L,\mathcal{T}) = H^1(U_L,\mathcal{T}).$$

In addition, setting $H \coloneqq \mathrm{Gal}(L/K)$, [26, Rk. 6.11] implies the existence for every natural number a of a commutative diagram of $\mathcal{Q}[\mathcal{G}_L]$-modules

$$\begin{array}{ccc}
{\bigwedge}^a_{\mathcal{Q}[\mathcal{G}_L]} H^1(U_L, \mathcal{Q} \otimes_{\mathcal{R}} \mathcal{T}) & \xrightarrow{\quad \wedge^a \mathrm{cores}_{L/K} \quad} & {\bigwedge}^a_{\mathcal{Q}[\mathcal{G}_L]} H^1(U_K, \mathcal{Q} \otimes_{\mathcal{R}} \mathcal{T}) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathcal{Q} \otimes_{\mathcal{R}} {\bigcap}^a_{\mathcal{R}[\mathcal{G}_L]} H^1(U_L, \mathcal{T}) & & \mathcal{Q} \otimes_{\mathcal{R}} {\bigcap}^a_{\mathcal{R}[\mathcal{G}_K]} H^1(U_K, \mathcal{T}) \\
& \searrow^{\cdot \mathrm{N}_H} \qquad \swarrow^{\nu^a_{L/K}}_{\simeq} & \\
& \mathcal{Q} \otimes_{\mathcal{R}} \big({\bigcap}^a_{\mathcal{R}[\mathcal{G}_L]} H^1(U_L, \mathcal{T}) \big)^H &
\end{array}$$

in which the horizontal map is induced by corestriction $H^1(U_L, \mathcal{T}) \to H^1(U_K, \mathcal{T})$ and both vertical isomorphisms are as in (4.8). This diagram implies that elements of $\mathrm{ES}^a(\mathfrak{F}_{\mathrm{rel}}(\mathcal{T}))$ coincide precisely with the Euler systems of rank a for the pair $(\mathcal{T}, \Omega)$ defined in [26, Def. 6.1].

4.2. Hypotheses on Galois representations

We assume to be given a pair of local complete Gorenstein rings

$$\mathcal{R} = \varprojlim_{i \in \mathbb{N}} \mathcal{R}_i \quad \text{and} \quad R = \varprojlim_{i \in \mathbb{N}} R_i$$

that arise as the respective inverse limits (as in § 2.1.2) of a family of rings $\mathcal{R}_i$ and R_i that are local, Gorenstein and of dimension zero. In particular, all transition morphisms $\mathcal{R}_{i+1} \to \mathcal{R}_i$ and $R_{i+1} \to R_i$ are assumed to be surjective.
We write $\mathcal{M}$ and M for the maximal ideals of $\mathcal{R}$ and R, and assume that the corresponding residue fields $\mathbb{K} \coloneqq \mathcal{R}/\mathcal{M}$ and $\mathbb{k} \coloneqq R/M$ are both finite and of characteristic p. We also adopt the convention that $\mathcal{R}_0 \coloneqq \mathbb{K}$ and $R_0 \coloneqq \mathbb{k}$.
The maximal ideals $\mathcal{M}_i$ and M_i of each $\mathcal{R}_i$ and R_i are the respective images of $\mathcal{M}$ and M under the canonical projections $\mathcal{R} \to \mathcal{R}_i$ and $R \to R_i$. In addition, the corresponding residue fields $\mathcal{R}_i/\mathcal{M}_i$ and R_i/M_i respectively identify with $\mathbb{K}$ and $\mathbb{k}$ and so are both independent of i.
We also assume to be given a morphism

$$\varrho \colon \mathcal{R} \to R$$

of local rings such that, for every natural number i, the diagram

$$\begin{array}{ccc}
\mathcal{R}_{i+1} & \xrightarrow{\varrho_{i+1}} & R_{i+1} \\
\downarrow & & \downarrow \\
\mathcal{R}_i & \xrightarrow{\varrho_i} & R_i
\end{array} \tag{4.10}$$

commutes. Here ϱ_i and ϱ_{i+1} denote the maps induced by ϱ, and the vertical arrows are the given transition maps that occur in the respective inverse limits.
In addition, we assume to be given a finitely generated free $\mathcal{R}$-module $\mathcal{T}$ that is endowed with a continuous action of G_k that is unramified outside a finite set of places $S_{\mathrm{ram}}(T)$. We set

$$T \coloneqq \mathcal{T} \otimes_{\mathcal{R}} R \qquad S \coloneqq S_{\mathrm{ram}}(T) \cup \Pi_k^p \cup \Pi_k^\infty$$

and, for every integer $i \geq 0$, also

$$\mathcal{T}_i \coloneqq \mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}_i \qquad T_i \coloneqq T \otimes_R R_i \qquad \overline{\mathcal{T}} \coloneqq \mathcal{T}_0 = \mathcal{T} \otimes_{\mathcal{R}} \mathbb{K} \qquad \overline{T} \coloneqq T_0 = T \otimes_R \mathbb{k}.$$

We write $\mathscr{F}$ for the Nekovář structure $\mathscr{F}_k$ on $\mathcal{T}$ fixed in the last section, and use the associated structures

$$\mathscr{F}_i \coloneqq \mathscr{F} \otimes_{\mathcal{R}} \mathcal{R}_i, \quad \mathcal{F}_i \coloneqq h(\mathscr{F}_i) \quad \text{and} \quad F_i \coloneqq h(\mathscr{F}_i \otimes_{\mathcal{R}_i} R_i)$$

on $\mathcal{T}_i$ and T_i. The Mazur–Rubin structures that are induced by $\mathcal{F}_i$ and F_i on $\overline{\mathcal{T}}$ and $\overline{T}$ (via the procedure described in Remark 3.23 (iv)) will then be denoted by $\overline{\mathcal{F}_i}$ and $\overline{F_i}$. We caution the reader that in general $\overline{\mathcal{F}_i}$ and $\overline{F_i}$ will be finer than $\mathcal{F}_0$ and F_0. For each i, we set

$l(i) := \max\{|\mathcal{R}_i|, |R_i|\}$. We also write $k(\mathcal{T}_i)$ for the minimal Galois extension of k for which $G_{k(\mathcal{T}_i)}$ acts trivially on $\mathcal{T}_i$. For each $i \in \mathbb{N}$, we then set

$$k_i := k(\mu_{l(i)}, (\mathcal{O}_k^\times)^{1/l(i)})k(1) \quad \text{and} \quad k_i(\mathcal{T}_i) := k(\mathcal{T}_i)k_i$$

and finally write k_∞ and $k(\mathcal{T})_\infty$ for the fields $\bigcup_{i\in\mathbb{N}} k_i$ and $\bigcup_{i\in\mathbb{N}} k_i(\mathcal{T}_i)$.
Given $i \in \mathbb{N}$ and an element $\tau \in G_{k_\infty}$, we define a subset of Π_k by setting

$$\mathcal{Q}_i = \mathcal{Q}(\tau, \mathcal{T}_i) := \{v \in \Pi_k \setminus S(\mathcal{T}_i) : \mathrm{Frob}_v \ \text{ is conjugate to } \ \tau \ \text{ in } \ \mathcal{G}_{k_i(\mathcal{T}_i)}\},$$

and write $\mathcal{N}_i := \mathcal{N}(\mathcal{Q}_i)$ for the set of square-free products of primes in $\mathcal{Q}_i$. We observe that there are decreasing filtrations

$$\mathcal{Q}_1 \supseteq \mathcal{Q}_2 \supseteq \dots \qquad \text{and} \qquad \mathcal{N}_1 \supseteq \mathcal{N}_2 \supseteq \dots,$$

and for each modulus $\mathfrak{n} \in \mathcal{N}_i$, we set

$$V(\mathfrak{n}) := \{\mathfrak{q} \in \mathcal{Q}_i : \mathfrak{q} \mid \mathfrak{n}\} = \{\mathfrak{q} \in \mathcal{Q}_1 : \mathfrak{q} \mid \mathfrak{n}\} \ \text{ and } \ \nu(\mathfrak{n}) := |V(\mathfrak{n})|.$$

For each natural number j with $j \geq i$, the Tate–Shafarevich group

$$Ш_{\overline{F_i},j}(k, \overline{T}) = Ш_{\overline{F_i}}(k, \overline{T}, \mathcal{Q}_j) := \ker\Big(H^1_{\overline{F_i}}(k, \overline{T}) \to \prod\nolimits_{v\in\mathcal{Q}_j} H^1(k_v, \overline{T})\Big)$$

is a submodule of $H^1_{\overline{F_i}(\mathfrak{n})}(k, \overline{T})$, where the modified Mazur–Rubin structure $\overline{F_i}(\mathfrak{n})$ is as defined in [74, Exam. 2.1.8] (see also [26, § 3.1.3]). Further, by using global duality sequences of the form (3.42), one checks that the integer

$$\chi(\overline{F_i}, j) := \dim_{\Bbbk}\big(H^1_{\overline{F_i}(\mathfrak{n})}(k, \overline{T})/Ш_{\overline{F_i},j}(k, \overline{T})\big) - \dim_{\Bbbk}\big(H^1_{\overline{F_i}^*(\mathfrak{n})}(k, \overline{T}^*(1))/Ш_{\overline{F_i}^*,j}(k, \overline{T}^*(1))\big) \tag{4.11}$$

is independent of $\mathfrak{n}$ (for details see § 6.1.3).
Before stating the relevant hypotheses, we make one further observation. Specifically, we note the maps in Lemma 3.48 combine with the general result of Lemma 2.32 to give natural maps

$$\mathrm{Tor}_1^{\mathcal{R}_{j'}}(X(\mathscr{F}_{j'}), R_{j'}) \to \mathrm{Tor}_1^{\mathcal{R}_j}(X(\mathscr{F}_j), R_j)$$

for integers j and j' with $j' \geq j$, and also a canonical isomorphism of $\mathcal{R}$-modules

$$\mathrm{Tor}_1^{\mathcal{R}}(X(\mathscr{F}), R) \cong \varprojlim\nolimits_{j\in\mathbb{N}} \mathrm{Tor}_1^{\mathcal{R}_j}(X(\mathscr{F}_j), R_j), \tag{4.12}$$

where the limit is taken with respect to the above maps for $j' \geq j$. For each natural number i, this gives rise to the following invariants J_i and $j(i)$ of $\mathscr{F}$.

(4.13) Definition. *For a natural number i we set*

$$J_i = J_i(\mathscr{F}) := \mathrm{im}\big(\mathrm{Tor}_1^{\mathcal{R}}(X(\mathscr{F}), R) \to \mathrm{Tor}_1^{\mathcal{R}_i}(X(\mathscr{F}_i), R_i)\big),$$

where the arrow denotes the map induced by (4.12). Then, since the group $\mathrm{Tor}_1^{\mathcal{R}_i}(X(\mathscr{F}_i), R_i)$ is finite, there exists a natural number $m \geq i$ such that J_i is equal to the image of the natural map $\mathrm{Tor}_1^{\mathcal{R}_m}(X(\mathscr{F}_m), R_m) \to \mathrm{Tor}_1^{\mathcal{R}_i}(X(\mathscr{F}_i), R_i)$. We define $j(i)$ to be the least possible value of such an m subject to the condition that the assignment $i \mapsto j(i)$ is an increasing function of i.

We can now finally state the hypotheses on $\mathcal{T}$ and $\mathscr{F}_k$ that will be used in our arguments.

(4.14) Hypotheses. The following conditions are satisfied.

- (i) The $\Bbbk[G_k]$-module $\overline{T}$ and the $\mathbb{K}[G_k]$-module $\overline{\mathcal{T}}$ are both irreducible.
- (ii) There exists an element τ of G_{k_∞} such that $\dim_{\mathbb{K}}(\overline{\mathcal{T}}/(\tau-1)\overline{\mathcal{T}}) = 1$.
- (ii*) If $p = 2$, then $\dim_{\mathbb{K}}(\overline{\mathcal{T}}) = 1$.
- (iii) $H^1(k(\mathcal{T})_\infty/k, \overline{T}^*(1)) = (0)$.
- (iv) If $p \in \{2, 3\}$, then the $\mathbb{Z}_p[G_k]$-modules $\overline{T} \oplus \overline{\mathcal{T}}$ and $\overline{T}^*(1) \oplus \overline{\mathcal{T}}^*(1)$ have no nonzero isomorphic subquotients.
- (v) $\mathscr{F}_k$ satisfies Hypothesis 3.44.

(vi) For every natural number i, one has $\chi(\overline{F_i}, j(i)) > 0$.

(vii) $\mathrm{Gal}(k(\mathcal{T})_\infty/k)$ is a compact p-adic analytic group.

(4.15) Remark. We clarify aspects of Hypotheses 4.14.

(i) It is clear that if $\mathcal{T}$ satisfies Hypothesis 4.14 (i) and (ii), respectively (ii)*, then so does $\mathcal{T}^*(1)$ (and with respect to the same element τ in (ii)).

(ii) The argument of [25, Lem. 3.8] implies that, under Hypothesis 4.14 (iii), the modules $H^1(k_j(\mathcal{T}_j)/k, \mathcal{T}_i^*(1))$ and $H^1(k_j(\mathcal{T}_j)/k, T_i^*(1))$ vanish for every $(i,j) \in \mathbb{N}^{\oplus 2}$ with $j \geq i$.

(iii) By the Jordan–Hölder Theorem (see the proof of Proposition 6.5 (ii) below), Hypothesis 4.14 (iv) is equivalent to the following condition: if both B_1 and B_2 denote either $\overline{T}$ or $\overline{\mathcal{T}}$, then B_1 and $B_2^*(1)$ have no nonzero isomorphic subquotients.

(iv) Sakamoto [96] has developed a theory of Kolyvagin systems in a case where $p = 3$ and Hypothesis 4.14 (iv) fails. It seems reasonable to believe that the approach of Sakamoto would also allow one to weaken Hypothesis 4.14 (iv) in our set-up.

(v) Hypothesis 4.14 (vi) is weaker than the 'cartesian' condition that originates with Mazur and Rubin [74] and is assumed throughout [26]. For example, subsequent analysis (in § 8.3.3 and § 8.4) will show that, in cases relevant to the study of Kato's 'generalised Iwasawa main conjecture', Hypothesis 4.14 (vi) is satisfied under wide-ranging, and natural, conditions.

(vi) Let T' be a continuous $\mathbb{Z}_p[\![G_k]\!]$-module that is free of rank t over $\mathbb{Z}_p$, with $k(T')$ the fixed field of k^c under the kernel of the induced homomorphism $G_k \to \mathrm{Aut}_{\mathbb{Z}_p}(T')$. Then $\mathrm{Gal}(k(T')/k)$ is isomorphic to a closed subgroup of $\mathrm{GL}_t(\mathbb{Z}_p)$ and so is a compact p-adic analytic group. In particular, if $\mathcal{R} = \mathbb{Z}_p[\![\mathrm{Gal}(L/k)]\!]$ for any compact p-adic analytic extension L/k and $\mathcal{T} = T' \otimes_{\mathbb{Z}_p} \mathcal{R}$, then $k(\mathcal{T})_\infty$ is the composite of the extensions $k(T')$, L and k_∞ of k and so Hypothesis 4.14 (vii) is valid. In addition, Hypothesis 4.14 (vii) can be omitted in any case in which certain Tate–Shafarevich groups are known to vanish (see Remark 6.37).

We also fix an abelian extension $\mathcal{K}$ of k in k^c and often assume the following hypothesis, in which we write $\Omega := \Omega(\mathcal{K})$ for the set of finite abelian extensions of k contained in $\mathcal{K}$.

(4.16) Hypotheses. The following conditions are satisfied.

(i) $\mathcal{K}$ is a pro-p extension that contains $k(\mathfrak{q})$ for all $\mathfrak{q} \in \mathcal{Q}_1$ and a $\mathbb{Z}_p$-power extension of k in which no finite place splits completely.

(ii) $\mathrm{Frob}_{\mathfrak{q}}^{p^k} - 1$ acts injectively on $\mathcal{T}$ for every $k \geq 0$ and all $\mathfrak{q} \in \mathcal{Q}_1$.

(4.17) Remark. In many situations there exist alternative conditions that can replace Hypothesis 4.16 (i). This aspect is discussed in more detail in [92, § 9.1].

4.3. Statement of the main result

We fix a family $\mathfrak{F} = (\mathscr{F}_K)_{K \in \Omega}$ of Nekovář structures satisfying Hypothesis 4.1 and set

$$\chi_{\mathfrak{F}} := \chi_{\mathcal{R}}(C(\mathscr{F}_k)) \in \mathbb{Z}.$$

We also assume to be given a non-zero free $\mathcal{R}$-module quotient Y of $\bigoplus_{\mathfrak{q} \in \Pi_k^\infty} H^0(k_{\mathfrak{q}}, \mathcal{T}^\vee(1))^\vee$. Such an $\mathcal{R}$-module is finitely generated and we set

$$r_Y := \mathrm{rk}_{\mathcal{R}}(Y) \quad \text{and} \quad r_{\mathfrak{F},Y} := r_Y + \chi_{\mathfrak{F}}. \tag{4.18}$$

We fix an $\mathcal{R}$-basis $b_\bullet$ of Y and use the surjective map in the central row of diagram (3.46) to regard Y as a quotient of $H^1(C(\mathscr{F}_k))$. We recall (from Proposition 3.45 (i) and (v)) that $C(\mathscr{F}_k)$ belongs to $D^{\mathrm{perf}}_{[0,1]}(\mathcal{R})$ and $H^0(C(\mathscr{F}_k)) = H^1_{\mathfrak{F}}(k, \mathcal{T})$. Hence, if $r_{\mathfrak{F},Y} > 0$, then the general construction of Lemma 2.37 (i) gives a map of $\mathcal{R}$-modules

$$\vartheta_{\mathscr{F}_k,Y} := \vartheta_{C(\mathscr{F}_k),b_\bullet} \colon \mathrm{Det}_{\mathcal{R}}(C(\mathscr{F}_k)) \to {\bigcap}_{\mathcal{R}}^{r_{\mathfrak{F},Y}} H^1_{\mathfrak{F}}(k, \mathcal{T}) \tag{4.19}$$

that depends only on $C(\mathscr{F}_k)$ and $b_\bullet$. (Note also that Remark 2.38 (i) implies a change of $b_\bullet$ will not affect the validity of any of the results stated subsequently and so, for clarity of exposition, we prefer to write $\vartheta_{\mathscr{F}_k,Y}$ in place of the more precise $\vartheta_{\mathscr{F}_k,b_\bullet}$).

We finally note that the $\mathcal{R}$-module $X(\mathscr{F}_k)$ is finitely-presented (by Lemma 3.48 and the fact $\mathcal{R}$ is Noetherian) and so, for each $a \in \mathbb{N}_0$, the Fitting ideal $\mathrm{Fitt}^a_{\mathcal{R}}(X(\mathscr{F}_k))$ is defined.

We can now state our main result concerning Euler systems for Nekovář structures.

(4.20) Theorem. *Let $\mathfrak{F} = (\mathscr{F}_K)_{K\in\Omega}$ be a family of Nekovář structures satisfying Hypothesis 4.1. Assume Hypotheses 4.14 and 4.16 are valid, and that the (non-zero) free $\mathcal{R}$-module Y is such that $r_{\mathfrak{F},Y} > 0$. Then, for each pair of elements x and y of $\mathrm{Fitt}^{r_Y}_{\mathcal{R}}(X(\mathscr{F}_k))$, there exists a natural number N that only depends on $\overline{\mathcal{T}}$ and has the following property: for every Euler system $c = (c_K)_K$ in $\mathrm{ES}^{r_{\mathfrak{F},Y}}(\mathfrak{F})$ and every prime ideal $\mathfrak{p}$ of $\mathcal{R}$ for which both*

(i) the map $\varrho_{\mathfrak{p}} \colon \mathcal{R}_{\mathfrak{p}} \to R_{\mathfrak{p}}$ induced by ϱ is nonzero and surjective, and

(ii) $\mathrm{Fitt}^0_R(\mathrm{Tor}^{\mathcal{R}}_1(X(\mathscr{F}_k), R))_{\mathfrak{p}} = R_{\mathfrak{p}}$,

one has $xy^N \cdot (c_k)_{\mathfrak{p}} \in y^N \cdot \vartheta_{\mathscr{F}_k,Y}(\mathrm{Det}_{\mathcal{R}}(C(\mathscr{F}_k)))_{\mathfrak{p}}$.

(4.21) Remark. The Nekovář structure $\mathscr{F}_k$ is restricted by the assumed existence of a non-zero free $\mathcal{R}$-module quotient Y of $\bigoplus_{\mathfrak{q}\in\Pi_k^\infty} H^0(k_{\mathfrak{q}}, \mathcal{T}^\vee(1))^\vee$ with $r_{\mathfrak{F},Y} > 0$. For example, if $\mathscr{F}_k$ is a Greenberg–Nekovář structure $\mathscr{F}((\mathcal{T}_{\mathfrak{q}}, j_{\mathfrak{q}})_{\mathfrak{q}\in S(\mathfrak{F}_k)})$ (as defined in Example 3.15 with $A = \mathcal{T}$) for which each sub-representation $\mathcal{T}_{\mathfrak{q}}$ is a free $\mathcal{R}$-module, then Proposition 3.45 (i) combines with Lemma 3.3 (ii) and (iii) to imply that

$$\chi_{\mathfrak{F}} = -\sum\nolimits_{\mathfrak{q}\in\Pi_k^p} [K_{\mathfrak{q}} : \mathbb{Q}_p] \cdot (\mathrm{rk}_{\mathcal{R}}(\mathcal{T}) - \mathrm{rk}_{\mathcal{R}}(\mathcal{T}_{\mathfrak{q}})).$$

Hence, in this case, there exists an $\mathcal{R}$-module Y of the required form with $r_{\mathfrak{F},Y} > 0$ if and only if the subrepresentations $\mathcal{T}_{\mathfrak{q}}$ for $\mathfrak{q} \in \Pi_k^p$ satisfy the following condition

$$\sum\nolimits_{\mathfrak{q}\in\Pi_k^p} [K_{\mathfrak{q}} : \mathbb{Q}_p] \cdot \mathrm{rk}_{\mathcal{R}}(\mathcal{T}_{\mathfrak{q}}) > [K : \mathbb{Q}] \cdot \mathrm{rk}_{\mathcal{R}}(\mathcal{T}) - \sum\nolimits_{\mathfrak{q}\in\Pi_k^\infty} \mathrm{rk}_{\mathcal{R}}(H^0(k_{\mathfrak{q}}, \mathcal{T}^\vee(1))^\vee).$$

Though technical in nature, Theorem 4.20 has significant advantages over the main results in the theory of Euler, Kolyvagin and Stark systems (in arbitrary rank) developed by Mazur and Rubin in [74, 75] and by Sakamoto, Sano and the first author in [26, 25]. Firstly, it is finer since its conclusion directly concerns the determinants of Selmer complexes rather than the Fitting ideals of Selmer groups. Secondly, it is more general in dealing with Euler systems relative to a wide class of Nekovář structures rather than only to the relaxed Nekovář structure. Thirdly, it is more widely applicable in arithmetic settings since several of the assumptions in Hypothesis 4.14 are weaker than the corresponding conditions imposed in [74, 75] and [26, 25] (see, for example, Remark 4.15(iv)). After a lengthy series of preliminary, and rather technical, results (some of which are possibly of independent interest), the proof of Theorem 4.20 will finally be obtained in §7. However, a reader who is more interested to understand how Theorem 4.20 can be applied in arithmetic settings rather than in the details of its proof, may prefer at this point to pass directly to the second part of the article (that starts in §8).

5. Kolyvagin systems I: modified Selmer complexes and derivatives

The theory of Kolyvagin systems was introduced by Mazur and Rubin in [74] and further developed by Sakamoto, Sano and the second author in [26], as a means of axiomatising Kolyvagin's construction of 'derivative classes' in [64]. In this, and the following, section we develop a version of this theory for the Selmer complexes arising from a family $\mathfrak{F}$ of Nekovář structures satisfying Hypothesis 4.1. These sections constitute the technical heart of our general approach. In them we fix a natural number i and, for clarity of exposition, we abbreviate the notation introduced in § 4.2 in the following way:

$$\begin{cases} \mathscr{F} := \mathscr{F}_k,\ \mathcal{Q} := \mathcal{Q}_i,\ \mathcal{N} := \mathcal{N}_i, \\ \mathbb{A} := \mathcal{R}_i,\ \mathcal{A} := \mathcal{T}_i,\ \mathcal{B} := \mathcal{A}^*(1),\ \tilde{\mathscr{F}} := \mathscr{F}_i,\ \tilde{\mathcal{F}} := h(\tilde{\mathscr{F}}), \\ \Lambda := R_i,\ A := T_i,\ B := A^*(1),\ \tilde{F} := h(\tilde{\mathscr{F}} \otimes_{\mathbb{A}} \Lambda), \\ \overline{A} := \mathcal{A} \otimes_{\mathbb{A}} \Bbbk = A \otimes_{\Lambda} \Bbbk = \overline{T},\ \overline{B} := \overline{A}^*(1),\ \overline{F} := \tilde{F}_{\overline{A}} \end{cases} \tag{5.1}$$

where $\tilde{F}_{\overline{A}}$ denotes the Mazur–Rubin structure on $\overline{A}$ induced by $\tilde{F}$ (cf. Example 3.23 (iv)). In particular with this notation, one has $\overline{\mathcal{T}} = \mathcal{A} \otimes_{\mathbb{A}} \mathbb{K}$ and $\overline{T} = \mathcal{A} \otimes_{\mathbb{A}} \Bbbk = A \otimes_{\Lambda} \Bbbk$.

5.1. Comparison maps

In this subsection, we construct several maps that play a key role in the sequel.
For $\mathfrak{q}$ in $\mathcal{Q}$, we write $k(\mathfrak{q})$ for the maximal p-extension of k in the ray class field modulo $\mathfrak{q}$ of k and note that, by [92, Lem. 4.1.2] (or [10, Lem. 2.1]), the group

$$G_{\mathfrak{q}} := \mathrm{Gal}(k(\mathfrak{q})/k(1))$$

is cyclic of order divisible by $l(i)$. Following Rubin [92, Def. 4.4.1], we then specify a generator $\sigma_{\mathfrak{q}}$ of $G_{\mathfrak{q}}$ as follows. We fix a topological generator ϖ of $\varprojlim_{n\in\mathbb{N}} \mu_{p^n}(\overline{\mathbb{Q}}) \cong \mathbb{Z}_p(1)$ and an embedding $\iota_{\mathfrak{q}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{k_{\mathfrak{q}}}$. This embedding induces an identification of $G_{\mathfrak{q}}$ with $\mathrm{Gal}(k(\mathfrak{q})_{\mathfrak{Q}}/k_{\mathfrak{q}})$, where $\mathfrak{Q}$ is the place of $k(\mathfrak{q})$ above $\mathfrak{q}$ specified by $\iota_{\mathfrak{q}}$. In particular, since the local reciprocity map $\mathrm{rec}_{\mathfrak{q}}$ identifies $\mathrm{Gal}(k(\mathfrak{q})_{\mathfrak{Q}}/k_{\mathfrak{q}})$ with a quotient of $\mu_{k_{\mathfrak{q}}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we specify the generator $\sigma_{\mathfrak{q}}$ to be the image of $\iota_{\mathfrak{q}}(\varpi)$ under the composite map

$$\varprojlim_{n\in\mathbb{N}} \mu_{p^n}(\overline{k_{\mathfrak{q}}}) \twoheadrightarrow \mu_{k_{\mathfrak{q}}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\mathrm{rec}_{\mathfrak{q}}} \mathrm{Gal}(k(\mathfrak{q})_{\mathfrak{Q}}/k_{\mathfrak{q}}) \cong G_{\mathfrak{q}}.$$

Similarly, for any $\mathfrak{n} \in \mathcal{N}$, we write $k(\mathfrak{n})$ for the compositum field $\prod_{\mathfrak{q}\in V(\mathfrak{n})} k(\mathfrak{q})$ and set $G_{\mathfrak{n}} := \mathrm{Gal}(k(\mathfrak{n})/k(1)) \cong \bigotimes_{\mathfrak{q}\in V(\mathfrak{n})} G_{\mathfrak{q}}$.
Next we note that the natural 'valuation' map $\mathrm{ord}_{\mathfrak{q}}$ induces an identification $k_{\mathfrak{q}}^{\mathrm{nr},\times}/(k_{\mathfrak{q}}^{\mathrm{nr},\times})^{l(i)} = \mathbb{Z}/l(i)\mathbb{Z}$. Hence, upon tensoring the canonical composite isomorphism

$$H^1(k_{\mathfrak{q}}^{\mathrm{nr}}, \mu_{l(i)}) \cong k_{\mathfrak{q}}^{\mathrm{nr},\times}/(k_{\mathfrak{q}}^{\mathrm{nr},\times})^{l(i)} = \mathbb{Z}/l(i)\mathbb{Z}$$

with $\mathcal{A}$, we obtain an identification of $H^1(k_{\mathfrak{q}}^{\mathrm{nr}}, \mathcal{A}(1))$ with $\mathcal{A}$. This combines with the generator ϖ of $\mathbb{Z}_p(1) = H^0(k_{\mathfrak{q}}^{\mathrm{nr}}, \mathbb{Z}_p(1))$ fixed above to give an isomorphism

$$\partial_{\mathfrak{q}} \colon H^1(k_{\mathfrak{q}}^{\mathrm{nr}}, \mathcal{A})^{G_{\kappa_{\mathfrak{q}}}} \overset{\cup\iota_{\mathfrak{q}}(\varpi)}{\cong} H^1(k_{\mathfrak{q}}^{\mathrm{nr}}, \mathcal{A}(1))^{G_{\kappa_{\mathfrak{q}}}} \xrightarrow{\simeq} \mathcal{A}^{\tau=1}.$$

The inflation-restriction sequence implies the existence of a canonical short exact sequence

$$0 \to H^1_f(k_{\mathfrak{q}}, \mathcal{A}) \to H^1(k_{\mathfrak{q}}, \mathcal{A}) \xrightarrow{\partial_{\mathfrak{q}}\circ\mathrm{res}_{\mathfrak{q}}} \mathcal{A}^{\tau=1} \to 0, \tag{5.2}$$

in which $\mathrm{res}_{\mathfrak{q}}$ denotes the (surjective) restriction map $H^1(k_{\mathfrak{q}}, \mathcal{A}) \to H^1(k_{\mathfrak{q}}^{\mathrm{nr}}, \mathcal{A})^{G_{\kappa_{\mathfrak{q}}}}$. One checks that $\partial_{\mathfrak{q}} \circ \mathrm{res}_{\mathfrak{q}}$ is explicitly given by evaluating a cocycle at $\sigma_{\mathfrak{q}}$, and hence agrees with the map used by Mazur and Rubin in [74, Lem. 1.2.1].
The next result relies on the validity of Hypothesis 4.14 (ii) and is essentially well-known (cf. [74, Lem. 1.2.3]). However, since it is important for us, we shall include a proof.

(5.3) Lemma. *There exists a canonical isomorphism of $\mathcal{R}_i$-modules $\mathcal{A}/(\tau-1) \to \mathcal{A}^{\tau=1}$.*

Proof. Each endomorphism f of the (free) $\mathbb{A}$-module $\mathcal{A}$ gives rise to a 'cofactor map' $c_f \colon \mathcal{A} \to \mathcal{A}$ that is uniquely characterised by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathbb{A}}\big(\bigwedge_{\mathbb{A}}^{\mathrm{rk}_{\mathbb{A}}(\mathcal{A})-1}\mathcal{A}, \bigwedge_{\mathbb{A}}^{\mathrm{rk}_{\mathbb{A}}(\mathcal{A})}\mathcal{A}\big) \\ \big\downarrow{\scriptstyle c_f} & & \big\downarrow{\scriptstyle h\mapsto h\circ(\bigwedge f)} \\ \mathcal{A} & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathbb{A}}\big(\bigwedge_{\mathbb{A}}^{\mathrm{rk}_{\mathbb{A}}(\mathcal{A})-1}\mathcal{A}, \bigwedge_{\mathbb{A}}^{\mathrm{rk}_{\mathbb{A}}(\mathcal{A})}\mathcal{A}\big), \end{array}$$

in which both horizontal arrows send $a \in \mathcal{A}$ to the map $x \mapsto a \wedge x$. (Thus, if we fix a Λ-basis of $\mathcal{A}$, then c_f is represented by the adjugate of the matrix representing f in this basis.) Then, as $f \circ c_f$ and $c_f \circ f$ both coincide with multiplication by $\det(f)$, there is a well-defined map

$$\mathcal{A}/f(\mathcal{A}) \xrightarrow{c_f} \ker\big(\mathcal{A}/\det(f)\mathcal{A} \xrightarrow{f} \mathcal{A}/\det(f)\mathcal{A}\big).$$

We now take f to be multiplication by $1-\tau$. Then $\det(f) = 0$ (as Hypothesis 4.14 (ii) implies $\ker(f)$ contains a non-zero divisor) and so the above construction gives a map of Λ-modules $c_f\colon \mathcal{A}/(\tau-1)\mathcal{A} \to \mathcal{A}^{\tau=1}$. It is enough to prove this map is bijective, or equivalently (as $\mathcal{A}$ is finite) surjective. Then, as Hypothesis 4.14 (ii) implies the natural map $\Bbbk \otimes_{\Lambda} \mathcal{A}^{\tau=1} \to \overline{A}^{\tau=1}$ is bijective, and $\Bbbk \otimes_{\Lambda} c_f$ identifies with the cofactor map of the reduction $\overline{f}\colon \overline{A} \to \overline{A}$ of f, Nakayama's lemma reduces us to proving $c_{\overline{f}}$ is bijective. But Hypothesis 4.14 (ii) implies the corank of $\overline{f}$ is one and hence that its adjugate matrix, and so also $c_{\overline{f}}$, is nontrivial. Since Hypothesis 4.14 (ii) also implies the $\Bbbk$-dimension of $\ker(\overline{f}) = \overline{A}^{\tau=1}$ is one, the obvious inclusion $\mathrm{im}(c_{\overline{f}}) \subseteq \ker(\overline{f})$ must be an equality, as required to prove the claim. □

We now fix an isomorphism of $\mathcal{R}_i$-modules $\mathcal{A}/(\tau-1) \cong \Lambda$ as in Hypothesis 4.14 (ii), and define a composite map of Λ-modules

$$\begin{aligned} v_{\mathfrak{q}}\colon H^1(k,\mathcal{A}) \otimes_{\mathbb{Z}} G_{\mathfrak{q}} \xrightarrow{\mathrm{loc}_{\mathfrak{q}}} H^1(k_{\mathfrak{q}},\mathcal{A}) \otimes_{\mathbb{Z}} G_{\mathfrak{q}} \overset{\mathrm{res}_{\mathfrak{q}}}{\twoheadrightarrow} H^1(k_{\mathfrak{q}}^{\mathrm{nr}},\mathcal{A})^{G_{\kappa_{\mathfrak{q}}}} \otimes_{\mathbb{Z}} G_{\mathfrak{q}} \\ \xrightarrow{\partial_{\mathfrak{q}}} \mathcal{A}^{\tau=1} \cong \mathcal{A}/(\tau-1) \cong \Lambda, \quad (5.4) \end{aligned}$$

in which the penultimate map is the isomorphism from Lemma 5.3.

We next observe that, since the inertia subgroup at $\mathfrak{q}$ acts trivially on $\mathcal{T}_i$ and $\mathrm{Frob}_{\mathfrak{q}}$ acts as τ, there exists a well-defined 'evaluation' map

$$H^1(k_{\mathfrak{q}},\mathcal{A}) \to \mathcal{A}/(\tau-1), \quad x \mapsto x(\mathrm{Frob}_{\mathfrak{q}}) + (\tau-1)\mathcal{T}_i. \quad (5.5)$$

This map is surjective and its kernel coincides with the 'transverse' cohomology group

$$H^1_{\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{A}) := H^1(G_{\mathfrak{q}},\mathcal{A}^{G_{k(\mathfrak{q})}}),$$

regarded as a submodule of $H^1(k_{\mathfrak{q}},\mathcal{A})$ via the inflation map (cf. the discussion of [76, § 1.2]). The 'finite-singular comparison map' is then defined to be the composite map of $\mathcal{R}_i$-modules

$$\psi^{\mathrm{fs}}_{\mathfrak{q}}\colon H^1(k,\mathcal{A}) \xrightarrow{\mathrm{loc}_{\mathfrak{q}}} H^1(k_{\mathfrak{q}},\mathcal{A}) \twoheadrightarrow \mathcal{A}/(\tau-1) \cong \Lambda \quad (5.6)$$

where the final map is the same isomorphism as fixed in (5.4).

5.2. Modified Selmer complexes

In this subsection we refine the 'modified Selmer structures' used in [74] by defining a corresponding family of 'modified Nekovář structures' and describing relations between their associated Selmer complexes.

We start by analysing the complex $\mathrm{R\Gamma}(k_{\mathfrak{q}},\mathcal{A})$ for each prime $\mathfrak{q} \in \mathcal{Q}$.

(5.7) Lemma. *For each prime $\mathfrak{q} \in \mathcal{Q}$, the following claims are valid.*

(i) The Λ-modules $H^0(k_{\mathfrak{q}},\mathcal{A}), H^1(k_{\mathfrak{q}},\mathcal{A})$ and $H^2(k_{\mathfrak{q}},\mathcal{A})$ are free of ranks $1,2$ and 1.

(ii) There exists a canonical isomorphism in $D^{\mathrm{perf}}(\Lambda)$

$$H^0(k_{\mathfrak{q}},\mathcal{A})[0] \oplus H^1(k_{\mathfrak{q}},\mathcal{A})[-1] \oplus H^2(k_{\mathfrak{q}},\mathcal{A})[-2] \cong \mathrm{R\Gamma}(k_{\mathfrak{q}},\mathcal{A}).$$

(iii) For $i \in \{0,1,2\}$ fix a projective Λ-submodule X_i of $H^i(k_{\mathfrak{q}},\mathcal{A})$ and write Y_i for the quotient $H^i(k_{\mathfrak{q}},\mathcal{A})/X_i$. Then the isomorphism in (ii) induces a canonical morphism in $D^{\mathrm{perf}}(\Lambda)$

$$\phi_{(X_0,X_1,X_2)}: X_0[0] \oplus X_1[-1] \oplus X_2[-2] \to \mathrm{R\Gamma}(k_{\mathfrak{q}},\mathcal{A}).$$

Further, local Tate duality identifies each module Y_i^* with a submodule of $H^{2-i}(k_{\mathfrak{q}},\mathcal{B})$ and also gives a canonical isomorphism in $D^{\mathrm{perf}}(\Lambda)$

$$\mathrm{Cone}\big(\phi_{(X_0,X_1,X_2)}\big)^*[-2] \cong Y_2^*[0] \oplus Y_1^*[-1] \oplus Y_0^*[-2].$$

Proof. For $\mathfrak{q} \in \mathcal{Q}$, one has $H^0(k_\mathfrak{q}, \mathcal{A}) = \mathcal{A}^{\tau=1}$ and $H^1_f(k_\mathfrak{q}, \mathcal{A}) = \mathcal{A}/(\tau-1)$ and so both $\mathbb{A}$-modules are free of rank one (see (5.4)). From the inflation-restriction sequence (5.2) it then follows that the $\mathbb{A}$-module $H^1(k_\mathfrak{q}, \mathcal{A})$ is free of rank 2. These observations combine with Proposition 3.3 (ii) to imply that the $\mathbb{A}$-module $H^2(k_\mathfrak{q}, \mathcal{A})$ has finite projective dimension and its class in $K_0(\mathbb{A})$ is equal to $[H^1(k_\mathfrak{q}, \mathcal{A})] - [H^0(k_\mathfrak{q}, \mathcal{A})] = [\mathbb{A}]$. Since $\mathbb{A}$ is both local and self-injective, it therefore follows that $H^2(k_\mathfrak{q}, \mathcal{A})$ is isomorphic to $\mathbb{A}$, as required to prove (i).
Fix $i \in \{0, 1, 2\}$. Then, since $H^i(k_\mathfrak{q}, \mathcal{A})$ is a projective $\mathbb{A}$-module, there exists a unique morphism $\theta_i : H^i(k_\mathfrak{q}, \mathcal{A})[-i] \to \mathrm{R\Gamma}(k_\mathfrak{q}, \mathcal{A})$ in $D(\mathbb{A})$ for which $H^i(\theta_i)$ is the identity map on $H^i(k_\mathfrak{q}, \mathcal{A})$. The direct sum morphism

$$\theta_0 \oplus \theta_1 \oplus \theta_2 \colon H^0(k_\mathfrak{q}, \mathcal{A})[0] \oplus H^1(k_\mathfrak{q}, \mathcal{A})[-1] \oplus H^2(k_\mathfrak{q}, \mathcal{A})[-2] \to \mathrm{R\Gamma}(k_\mathfrak{q}, \mathcal{A})$$

is then an isomorphism in $D(\mathbb{A})$, thereby proving (ii).
For X_0, X_1, X_2 as in (iii), the isomorphism in (ii) gives rise to an exact triangle in $D^{\mathrm{perf}}(\mathbb{A})$

$$X_0[0] \oplus X_1[-1] \oplus X_2[-2] \xrightarrow{\phi_{(X_0,X_1,X_2)}} \mathrm{R\Gamma}(k_\mathfrak{q}, \mathcal{A}) \to Y_0[0] \oplus Y_1[-1] \oplus Y_2[-2] \to \cdot.$$

Claim (iii) follows directly from this triangle. □

For any collection $\mathfrak{a}, \mathfrak{b}, \mathfrak{n}$ of pairwise coprime moduli in $\mathcal{N}$ for which $V(\mathfrak{abn}) \cap S(\tilde{\mathscr{F}}) = \varnothing$, we now define a 'modified Nekovář structure' $\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n})$ on $\mathcal{A}$ in the following way: we set

$$S(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n})) \coloneqq S(\tilde{\mathscr{F}}) \cup V(\mathfrak{abn})$$

and assign the condition at $\mathfrak{q} \in \Pi_k$ to be

$$\big(\mathrm{R\Gamma}_{\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n})}(k_\mathfrak{q}, \mathcal{A}), \theta_{\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}),\mathfrak{q}}\big) \coloneqq \begin{cases} \big(\mathrm{R\Gamma}_{\tilde{\mathscr{F}}}(k_\mathfrak{q}, \mathcal{A}), \theta_{\tilde{\mathscr{F}},\mathfrak{q}}\big) & \text{if } \mathfrak{q} \notin V(\mathfrak{abn}), \\ \big(H^0(k_\mathfrak{q}, \mathcal{A})[0], \phi_{\mathcal{A}.\mathfrak{q}}\big) & \text{if } \mathfrak{q} \in V(\mathfrak{a}), \\ \big(\mathrm{R\Gamma}(k_\mathfrak{q}, \mathcal{A}), \mathrm{id}\big) & \text{if } \mathfrak{q} \in V(\mathfrak{b}), \\ \big(H^0(k_\mathfrak{q}, \mathcal{A})[0] \oplus H^1_{\mathrm{tr}}(k_\mathfrak{q}, \mathcal{A})[-1], \phi_{\mathcal{A},\mathfrak{q}}\big) & \text{if } \mathfrak{q} \in V(\mathfrak{n}). \end{cases}$$

with $\phi_{\mathcal{A},\mathfrak{q}} \coloneqq \phi_{(H^0(k_\mathfrak{q},\mathcal{A}),(0),(0))}$ for $\mathfrak{q} \in V(\mathfrak{a})$ and $\phi_{\mathcal{A},\mathfrak{q}} \coloneqq \phi_{(H^0(k_\mathfrak{q},\mathcal{A}),H^1_{\mathrm{tr}}(k_\mathfrak{q},\mathcal{A}),(0))})$ for $\mathfrak{q} \in V(\mathfrak{n})$.

(5.8) Remark. If any of the moduli $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{n}$ are equal to the empty product of primes, then, for convenience, we will omit them from the notation $\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n})$. In particular, with this convention, the structure $\tilde{\mathscr{F}}^\mathfrak{b}$ coincides with the modification $\tilde{\mathscr{F}}^{V(\mathfrak{b})}$ of $\tilde{\mathscr{F}}$ defined in Example 3.16. Further, for all moduli $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{n}$ as above, an explicit check shows that the induced Mazur–Rubin structure $h(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}))$ agrees with the modification $\tilde{\mathcal{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n})$ of $\tilde{\mathcal{F}} = h(\tilde{\mathscr{F}})$ that is introduced by Mazur and Rubin in [74, Exam. 2.1.8] and used extensively in [26].

The following consequence of Proposition 3.45 will play a key role in our approach.

(5.9) Proposition. *Assume $\Pi_k^\infty = \varnothing$ if $p = 2$, that $\mathcal{A}$ is a finitely generated free $\mathbb{A}$-module and that $\tilde{\mathscr{F}}$ satisfies Hypothesis 3.44. Then, as $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{n}$ range over all pairwise coprime moduli in $\mathcal{N}$ for which $V(\mathfrak{abn}) \cap S(\tilde{\mathscr{F}}) = \varnothing$, the following claims are valid.*

(i) $C(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}))$ belongs to $D^{\mathrm{perf}}(\mathbb{A})$ and is such that, in $K_0(\mathbb{A})$, one has

$$\begin{aligned} \chi_\mathbb{A}\big(C(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}))\big) &= \chi_\mathbb{A}\big(C(\tilde{\mathscr{F}})\big) - \nu(\mathfrak{a}) \cdot [\mathbb{A}] \\ &= -\nu(\mathfrak{a}) \cdot [\mathbb{A}] + \sum\nolimits_{\mathfrak{q} \in S(\tilde{\mathscr{F}}) \setminus \Pi_k^\infty} \chi_\mathbb{A}(\mathrm{R\Gamma}_{\tilde{\mathscr{F}}^*}(k_\mathfrak{q}, B)). \end{aligned}$$

(ii) For any morphism $\mathbb{A} \to \mathbb{A}'$ of rings satisfying (2.5), there exists a natural isomorphism $C(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n})) \otimes^{\mathbb{L}}_\mathbb{A} \mathbb{A}' \cong C((\tilde{\mathscr{F}} \otimes_\mathbb{A} \mathbb{A}')^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}))$ in $D^{\mathrm{perf}}(\mathbb{A}')$.

(iii) $C(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}))$ is acyclic outside degrees 0 and 1. In addition, $H^0(C(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}))) = H^1_{\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n})}(k, \mathcal{A})$ and there exists a canonical short exact sequence of $\mathbb{A}$-modules

$$0 \to H^1_{(\tilde{\mathcal{F}}^*)^\mathfrak{a}_\mathfrak{b}(\mathfrak{n})}(k, \mathcal{B})^* \to H^1(C(\tilde{\mathscr{F}}^\mathfrak{b}_\mathfrak{a}(\mathfrak{n}))) \to X(\tilde{\mathscr{F}}) \oplus \bigoplus\nolimits_{\mathfrak{q} \in V(\mathfrak{b})} H^0(k_\mathfrak{q}, \mathcal{B})^* \to 0.$$

(iv) *For all divisors* $\mathfrak{n}$ *of* $\mathfrak{m}$ *there are canonical exact triangles in* $D^{\mathrm{perf}}(\mathbb{\Lambda})$

$$C(\tilde{\mathscr{F}}^{\mathfrak{n}}) \to C(\tilde{\mathscr{F}}^{\mathfrak{m}}) \to \bigoplus\nolimits_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})}\big(H^1_{/f}(k_{\mathfrak{q}},\mathcal{A})[0]\oplus H^2(k_{\mathfrak{q}},\mathcal{A})[-1]\big) \to \cdot \tag{5.10}$$

$$C(\tilde{\mathscr{F}}(\mathfrak{m})) \to C(\tilde{\mathscr{F}}^{\mathfrak{m}}) \to \bigoplus\nolimits_{\mathfrak{q}\in V(\mathfrak{m})}\big(H^1_{/\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{A})[0]\oplus H^2(k_{\mathfrak{q}},\mathcal{A})[-1]\big) \to \cdot \tag{5.11}$$

$$C(\tilde{\mathscr{F}}_{\mathfrak{n}}(\mathfrak{m}/\mathfrak{n})) \to C(\tilde{\mathscr{F}}(\mathfrak{m})) \to \bigoplus\nolimits_{\mathfrak{q}\in V(\mathfrak{n})}H^1_{\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{A})[0] \to \cdot \tag{5.12}$$

$$C(\tilde{\mathscr{F}}_{\mathfrak{a}}(\mathfrak{m})) \to C(\tilde{\mathscr{F}}(\mathfrak{m})) \to \bigoplus\nolimits_{\mathfrak{q}\in V(\mathfrak{a})}H^1_{f}(k_{\mathfrak{q}},\mathcal{A})[0] \to \cdot \tag{5.13}$$

(v) *Let* Φ *denote either the Nekovář structure* $\tilde{\mathscr{F}}$ *or Mazur–Rubin structure* $\tilde{\mathcal{F}} = h(\tilde{\mathscr{F}})$ *on* $\mathcal{A}$. *Then the cohomology sequences of the exact triangles in (iv) combine with the descriptions of cohomology in (iii) to induce the following exact sequences of* $\mathbb{\Lambda}$*-modules*

$$H^1_{\Phi^{\mathfrak{n}}}(k,\mathcal{A}) \hookrightarrow H^1_{\Phi^{\mathfrak{m}}}(k,\mathcal{A}) \xrightarrow{(\hat{v}_{\mathfrak{q}})_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})}} \mathbb{\Lambda}^{\nu(\mathfrak{m}/\mathfrak{n})} \to H^1_{(\tilde{\mathcal{F}}^*)_{\mathfrak{n}}}(k,\mathcal{B})^* \twoheadrightarrow H^1_{(\tilde{\mathcal{F}}^*)_{\mathfrak{m}}}(k,\mathcal{B})^* \tag{5.14}$$

$$H^1_{\Phi(\mathfrak{m})}(k,\mathcal{A}) \hookrightarrow H^1_{\Phi^{\mathfrak{m}}}(k,\mathcal{A}) \xrightarrow{(\hat{\psi}^{\mathrm{fs}}_{\mathfrak{q}})_{\mathfrak{q}\in V(\mathfrak{m})}} \mathbb{\Lambda}^{\nu(\mathfrak{m})} \to H^1_{\tilde{\mathcal{F}}^*(\mathfrak{m})}(k,\mathcal{B})^* \twoheadrightarrow H^1_{\tilde{\mathcal{F}}^*_{\mathfrak{m}}}(k,\mathcal{B})^* \tag{5.15}$$

$$H^1_{\Phi_{\mathfrak{n}}(\mathfrak{m}/\mathfrak{n})}(k,\mathcal{A}) \hookrightarrow H^1_{\Phi(\mathfrak{m})}(k,\mathcal{A}) \xrightarrow{(\check{v}_{\mathfrak{q}})_{\mathfrak{q}\in V(\mathfrak{n})}} \mathbb{\Lambda}^{\nu(\mathfrak{n})} \to H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{n}}(\mathfrak{m}/\mathfrak{n})}(k,\mathcal{B})^* \twoheadrightarrow H^1_{\tilde{\mathcal{F}}^*(\mathfrak{m})}(k,\mathcal{B})^* \tag{5.16}$$

$$H^1_{\Phi_{\mathfrak{a}}(\mathfrak{m})}(k,\mathcal{A}) \hookrightarrow H^1_{\Phi(\mathfrak{m})}(k,\mathcal{A}) \xrightarrow{(\check{\psi}^{\mathrm{fs}}_{\mathfrak{q}})_{\mathfrak{q}\in V(\mathfrak{a})}} \mathbb{\Lambda}^{\nu(\mathfrak{a})} \to H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{a}}(\mathfrak{m})}(k,\mathcal{B})^* \twoheadrightarrow H^1_{\tilde{\mathcal{F}}^*(\mathfrak{m})}(k,\mathcal{B})^* \tag{5.17}$$

If $\Phi = \tilde{\mathscr{F}}$, *we write* $\hat{v}_{\mathfrak{q}}$ *and* $\hat{\psi}^{\mathrm{fs}}_{\mathfrak{q}}$, *respectively* $\check{v}_{\mathfrak{q}}$ *and* $\check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}$, *for the composites of* $v_{\mathfrak{q}}$ *and* $\psi^{\mathrm{fs}}_{\mathfrak{q}}$ *with the canonical maps* $H^1_{\tilde{\mathscr{F}}^{\mathfrak{m}}}(k,\mathcal{A}) \to H^1_{\tilde{\mathcal{F}}^{\mathfrak{m}}}(k,\mathcal{A})$, *respectively* $H^1_{\tilde{\mathscr{F}}(\mathfrak{m})}(k,\mathcal{A}) \to H^1_{\tilde{\mathcal{F}}(\mathfrak{m})}(k,\mathcal{A})$. *If* $\Phi = \tilde{\mathcal{F}}$, *both* $\hat{v}_{\mathfrak{q}}$ *and* $\check{v}_{\mathfrak{q}}$ *denote* $v_{\mathfrak{q}}$ *and both* $\hat{\psi}^{\mathrm{fs}}_{\mathfrak{q}}$ *and* $\check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}$ *denote* $\psi^{\mathrm{fs}}_{\mathfrak{q}}$.

Proof. We write $\mathscr{D}$ for the dual Nekovář structure $\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})^*$ on $\mathcal{B}$. Then $S(\mathscr{D}) = S(\tilde{\mathscr{F}}) \cup V(\mathfrak{abn})$ and Lemma 5.7 (iii) implies that the local condition for $\mathscr{D}$ at $\mathfrak{q}\in\Pi_k$ is

$$\big(\mathrm{R\Gamma}_{\mathscr{D}}(k_{\mathfrak{q}},\mathcal{A}),\theta_{\mathscr{D},\mathfrak{q}}\big) \coloneqq \begin{cases} \big(\mathrm{R\Gamma}_{\tilde{\mathscr{F}}^*}(k_{\mathfrak{q}},\mathcal{B}),\theta_{\tilde{\mathscr{F}}^*,\mathfrak{q}}\big) & \text{if } \mathfrak{q}\notin V(\mathfrak{abn}),\\ (H^0(k_{\mathfrak{q}},\mathcal{B})[0]\oplus H^1(k_{\mathfrak{q}},\mathcal{B})[-1],\theta_{\mathscr{D},\mathfrak{q}}) & \text{if } \mathfrak{q}\in V(\mathfrak{a}),\\ (0,0) & \text{if } \mathfrak{q}\in V(\mathfrak{b}),\\ (H^0(k_{\mathfrak{q}},\mathcal{B})[0]\oplus H^1_{\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{B})[-1],\theta_{\mathscr{D},\mathfrak{q}}) & \text{if } \mathfrak{q}\in V(\mathfrak{n}),\end{cases} \tag{5.18}$$

with

$$\theta_{\mathscr{D},\mathfrak{q}} \coloneqq \begin{cases}\phi_{(H^0(k_{\mathfrak{q}},\mathcal{B}),H^1(k_{\mathfrak{q}},\mathcal{B}),(0))}, & \text{if } \mathfrak{q}\in V(\mathfrak{a}),\\ \phi_{(H^0(k_{\mathfrak{q}},\mathcal{B}),H^0_{\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{B}),(0))}, & \text{if } \mathfrak{q}\in V(\mathfrak{n}).\end{cases}$$

The fact $\tilde{\mathscr{F}}$ satisfies Hypothesis 3.44 implies that the same is true for $\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})$. In particular, the result of Proposition 3.45 (i) for $\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})$ directly implies that $C(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}))$ belongs to $D^{\mathrm{perf}}(\mathbb{\Lambda})$ and that its Euler characteristic in $K_0(\mathbb{\Lambda})$ is equal to

$$\begin{aligned}\textstyle\sum_{\mathfrak{q}\in S(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}))\setminus\Pi_k^\infty}\chi_R\big(\mathrm{R\Gamma}_{\mathscr{D}}(k_{\mathfrak{q}},\mathcal{B})\big) &= \textstyle\sum_{\mathfrak{q}\in S(\tilde{\mathscr{F}})\setminus\Pi_k^\infty}\chi_{\mathbb{\Lambda}}\big(\mathrm{R\Gamma}_{\tilde{\mathscr{F}}^*}(k_{\mathfrak{q}},\mathcal{B})\big)\\ &\quad + \textstyle\sum_{\mathfrak{q}\in V(\mathfrak{a})}\chi_{\mathbb{\Lambda}}\big(H^0(k_{\mathfrak{q}},B)[0]\oplus H^1(k_{\mathfrak{q}},\mathcal{B})[-1]\big)\\ &\quad + \textstyle\sum_{\mathfrak{q}\in V(\mathfrak{n})}\chi_{\mathbb{\Lambda}}\big(H^0(k_{\mathfrak{q}},B)[0]\oplus H^1_{\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{B})[-1]\big)\\ &= \textstyle\sum_{\mathfrak{q}\in S(\tilde{\mathscr{F}})\setminus\Pi_k^\infty}\chi_{\mathbb{\Lambda}}(\mathrm{R\Gamma}_{\tilde{\mathscr{F}}^*}(k_{\mathfrak{q}},\mathcal{B})) - \sum_{\mathfrak{q}\in V(\mathfrak{a})}[\mathbb{\Lambda}]\\ &= \chi_{\mathbb{\Lambda}}\big(C(\tilde{\mathscr{F}})\big) - \nu(\mathfrak{a})\cdot[\mathbb{\Lambda}].\end{aligned}$$

Here the first equality follows directly from the explicit description (5.18), the second from Lemma 5.7 (i) and the fact $H^1_{\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{B})$ is a free $\mathbb{\Lambda}$-module of rank 1 and the last follows directly from Proposition 3.45 (i). This proves (i).

To prove (ii) we recall from Proposition 3.3 (vi) that, for each $\mathfrak{q} \in V(\mathfrak{abn})$, there exists a natural isomorphism $\mathrm{R}\Gamma(k_\mathfrak{q}, \mathcal{A}) \otimes_{\mathbb{A}} \mathbb{A}' \cong \mathrm{R}\Gamma(k_\mathfrak{q}, \mathcal{A} \otimes_{\mathbb{A}} \mathbb{A}')$ in $D(\mathbb{A}')$. In conjunction with Lemma 5.7 (ii), this isomorphism induces, in each degree i, an isomorphism of $\mathbb{A}'$-modules $H^i(k_\mathfrak{q}, \mathcal{A}) \otimes_{\mathbb{A}} \mathbb{A}' \cong H^i(k_\mathfrak{q}, \mathcal{A} \otimes_{\mathbb{A}} \mathbb{A}')$. Further, for $\mathfrak{q} \in V(\mathfrak{n})$, the isomorphism $H^1(k_\mathfrak{q}, \mathcal{A}) \otimes_{\mathbb{A}} \mathbb{A}' \cong H^1(k_\mathfrak{q}, \mathcal{A} \otimes_{\mathbb{A}} \mathbb{A}')$ combines with the canonical isomorphisms $\big(\mathcal{A}/(\tau-1)\mathcal{A}\big) \otimes_{\mathbb{A}} \mathbb{A}' \cong (\mathcal{A} \otimes_{\mathbb{A}} \mathbb{A}')/(\tau - 1)$ to imply the image under $- \otimes_{\mathbb{A}} \mathbb{A}'$ of the map (5.5) for $\mathcal{A}$ is equal to the corresponding map for $\mathcal{A} \otimes_{\mathbb{A}} \mathbb{A}'$ and hence that there exists a canonical isomorphism $H^1_{\mathrm{tr}}(k_\mathfrak{q}, \mathcal{A}) \otimes_{\mathbb{A}} \mathbb{A}' \cong H^1_{\mathrm{tr}}(k_\mathfrak{q}, \mathcal{A} \otimes_{\mathbb{A}} \mathbb{A}')$ with respect to which $\phi_{\mathcal{A},\mathfrak{q}} \otimes_{\mathbb{A}} \mathbb{A}'$ identifies with $\phi_{\mathcal{A}\otimes_{\mathbb{A}}\mathbb{A}',\mathfrak{q}}$. Taken together, these observations imply the induced structure $\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}) \otimes_{\mathbb{A}} \mathbb{A}'$ identifies with $(\tilde{\mathscr{F}} \otimes_{\mathbb{A}} \mathbb{A}')^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})$ and so the isomorphism in (ii) follows from Proposition 3.45 (ii) for $\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})$.
Regarding (iii), we note Proposition 3.45 (v) for $\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})$ directly implies that $C(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}))$ is acyclic outside degrees 0 and 1 and $H^0(C(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}))) = H^1_{\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})}(k, \mathcal{A})$. From Remark 5.8, we also know that the dual Mazur–Rubin structure $h(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}))^*$ on B is equal to $\tilde{\mathcal{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})^* = (\tilde{\mathcal{F}}^*)^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{n})$. In addition, since $\tilde{\mathscr{F}}$ is assumed to validate Hypothesis 3.44 (b) (i) and (d), one has $H^0(\mathrm{R}\Gamma_{\mathscr{F}^*}(k_{\mathfrak{q}_1}, B)) = (0)$ and so the map $\lambda^0_{S(\tilde{\mathscr{F}})\cup V(\mathfrak{a})}(\mathscr{D})$ from (3.26) is injective. From Lemma 3.27 (ii) with $\mathscr{F}, S$ and S' taken to be $\mathscr{D}, S(\mathscr{D}) = S(\tilde{\mathscr{F}}) \cup V(\mathfrak{abn})$ and $V(\mathfrak{bn})$, one therefore obtains a short exact sequence of $\mathbb{A}$-modules

$$0 \to X_{S(\tilde{\mathscr{F}})\cup V(\mathfrak{a})}(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})) \to X_{S(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}))}(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})) \to \bigoplus\nolimits_{\mathfrak{q}\in V(\mathfrak{bn})} \left(\frac{H^0(k_\mathfrak{q}, B)}{\mathrm{im}(H^0(\theta_{\mathscr{D},\mathfrak{q}}))}\right)^* \to 0.$$

Further, from (5.18) one checks that $H^0(\theta_{\mathscr{D},\mathfrak{q}})$ is the zero map for $\mathfrak{q} \in V(\mathfrak{b})$ and is surjective for $\mathfrak{q} \in V(\mathfrak{an})$, and so this sequence is equivalent to an exact sequence

$$0 \to X(\tilde{\mathscr{F}}) \to X(\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})) \to \bigoplus\nolimits_{\mathfrak{q}\in V(\mathfrak{b})} H^0(k_\mathfrak{q}, \mathcal{B})^* \to 0.$$

This sequence splits since, for each $\mathfrak{q} \in V(\mathfrak{b})$, the $\mathbb{A}$-module $H^0(k_\mathfrak{q}, \mathcal{B})^*$ is free. Given these observations, the exact sequence in (iii) is now obtained from the lower row of the diagram (3.46) with $\mathscr{F}$ taken to be $\tilde{\mathscr{F}}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})$.
The exact triangles in (iv) are all derived from Lemma 3.12 (iii). In the first case, the descriptions (5.18) imply $(\tilde{\mathscr{F}}^{\mathfrak{m}})^* \le (\tilde{\mathscr{F}}^{\mathfrak{n}})^*$ and also, for $\mathfrak{q} \in V(\mathfrak{m}/\mathfrak{n})$, that $\mathrm{R}\Gamma_{(\tilde{\mathscr{F}}^{\mathfrak{n}})^*/(\tilde{\mathscr{F}}^{\mathfrak{m}})^*}(k_\mathfrak{q}, \mathcal{B})$ is isomorphic to $H^0(k_\mathfrak{q}, \mathcal{B})[0] \oplus H^1_f(k_\mathfrak{q}, \mathcal{B})[-1]$. In this case, therefore, Lemma 3.12 (iii) gives an exact triangle

$$\mathrm{R}\Gamma_{(\tilde{\mathscr{F}}^{\mathfrak{m}})^*}(k, \mathcal{B}) \to \mathrm{R}\Gamma_{(\tilde{\mathscr{F}}^{\mathfrak{n}})^*}(k, \mathcal{B}) \to \bigoplus\nolimits_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})} \big(H^0(k_\mathfrak{q}, \mathcal{B})[0] \oplus H^1_f(k_\mathfrak{q}, \mathcal{B})[-1]\big) \to \cdot$$

and (5.10) is directly derived from the image of this triangle under the exact functor $(-)^*[-2]$.
In a similar way, (5.18) implies $(\tilde{\mathscr{F}}^{\mathfrak{m}})^* \le \tilde{\mathscr{F}}(\mathfrak{m})^*$ and that, for each $\mathfrak{q} \in V(\mathfrak{m})$, there exists a natural isomorphism in $D^{\mathrm{perf}}(\mathbb{A})$

$$\begin{aligned}\mathrm{R}\Gamma_{\tilde{\mathscr{F}}(\mathfrak{m})^*/(\tilde{\mathscr{F}}^{\mathfrak{m}})^*}(k_\mathfrak{q}, \mathcal{B})^*[-2] &\cong \big(H^0(k_\mathfrak{q}, \mathcal{B})[0] \oplus H^1_{\mathrm{tr}}(k_\mathfrak{q}, \mathcal{B})[-1]\big)^*[-2] \\ &\cong H^1_{/\mathrm{tr}}(k_\mathfrak{q}, \mathcal{A})[-1] \oplus H^2(k_\mathfrak{q}, \mathcal{A})[-2].\end{aligned}$$

Given these facts, the exact triangle (5.11) is directly derived from the image under $(-)^*[-2]$ of the exact triangle in Lemma 3.12 (iii) with $(\mathscr{F}', \mathscr{F})$ taken to be $((\tilde{\mathscr{F}}^{\mathfrak{m}})^*, \tilde{\mathscr{F}}(\mathfrak{m})^*)$.
We next use (5.18) to check that $\tilde{\mathscr{F}}(\mathfrak{m})^* \le \tilde{\mathscr{F}}_{\mathfrak{n}}(\mathfrak{m}/\mathfrak{n})^*$, with $\mathrm{R}\Gamma_{\tilde{\mathscr{F}}_{\mathfrak{n}}(\mathfrak{m}/\mathfrak{n})^*/\tilde{\mathscr{F}}(\mathfrak{m})^*}(k_\mathfrak{q}, \mathcal{B})$ acyclic for each $\mathfrak{q} \notin V(\mathfrak{n})$ and naturally isomorphic to $H^1_{/\mathrm{tr}}(k_\mathfrak{q}, \mathcal{B})[-1]$ for each $\mathfrak{q} \in V(\mathfrak{n})$. Given these facts, the exact triangle (5.12) is derived from the image under $(-)^*[-2]$ of the triangle in Lemma 3.12 (iii) with $(\mathscr{F}', \mathscr{F})$ taken to be $(\tilde{\mathscr{F}}(\mathfrak{m})^*, \tilde{\mathscr{F}}_{\mathfrak{n}}(\mathfrak{m}/\mathfrak{n})^*)$.
Finally, to complete the proof of (iv), we note that (5.18) implies $\tilde{\mathscr{F}}(\mathfrak{m})^* \le \tilde{\mathscr{F}}_{\mathfrak{a}}(\mathfrak{m})^*$ and that $\mathrm{R}\Gamma_{\tilde{\mathscr{F}}_{\mathfrak{a}}(\mathfrak{m})^*/\tilde{\mathscr{F}}(\mathfrak{m})^*}(k_\mathfrak{q}, \mathcal{B})$ is acyclic for each $\mathfrak{q} \notin V(\mathfrak{a})$ and identifies with $H^1_{/f}(k_\mathfrak{q}, B)[-1]$ for each $\mathfrak{q} \in V(\mathfrak{a})$. These observations imply that the exact triangle (5.13) is directly derived from the image under $(-)^*[-2]$ of the exact triangle in Lemma 3.12 (iii) with $(\mathscr{F}', \mathscr{F})$ taken to be $(\tilde{\mathscr{F}}(\mathfrak{m})^*, \tilde{\mathscr{F}}_{\mathfrak{a}}(\mathfrak{m})^*)$.

Each of the exact sequences in (v) is derived from the long exact cohomology sequence of the corresponding exact triangle in (iv). For example, the long exact cohomology sequence of (5.10) combines with the descriptions of cohomology in (iii) to give the following variant of the exact commutative diagram in Theorem 3.43 (ii)

$$\begin{array}{ccccccccc}
H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A}) & \hookrightarrow & H^1_{\tilde{\mathscr{F}}^{\mathfrak{m}}}(k,\mathcal{A}) & \longrightarrow & \bigoplus_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})} H^1_{/f}(k_{\mathfrak{q}},\mathcal{A}) & & & & \\
\downarrow & & \downarrow & & \cong \downarrow (\alpha_{\mathfrak{q}})_{\mathfrak{q}} & & & & \\
H^1_{\tilde{\mathcal{F}}^{\mathfrak{n}}}(k,\mathcal{A}) & \hookrightarrow & H^1_{\tilde{\mathcal{F}}^{\mathfrak{m}}}(k,\mathcal{A}) & \xrightarrow{(v_{\mathfrak{q}})_{\mathfrak{q}}} & \mathbb{A}^{\nu(\mathfrak{m}/\mathfrak{n})} & \longrightarrow & H^1_{(\tilde{\mathcal{F}}^*)_{\mathfrak{n}}}(k,\mathcal{B})^* & \twoheadrightarrow & H^1_{(\tilde{\mathcal{F}}^*)_{\mathfrak{m}}}(k,\mathcal{B})^*.
\end{array}$$

Here the first two vertical maps are the canonical projections from (3.28) and each $\alpha_{\mathfrak{q}}$ denotes the isomorphism $H^1_{/f}(k_{\mathfrak{q}},\mathcal{A}) \cong \mathbb{A}$ induced by the composite map

$$H^1(k_{\mathfrak{q}},\mathcal{A}) \xrightarrow{\text{res}} H^1(k_{\mathfrak{q}}^{\text{nr}},\mathcal{A})^{G_{\kappa_{\mathfrak{q}}}} \xrightarrow{\partial_{\mathfrak{q}}} \mathcal{A}^{\tau=1} \cong \mathcal{A}/(\tau-1) \cong \mathbb{A} \tag{5.19}$$

that occurs in the definition (5.4) of $v_{\mathfrak{q}}$. Taking account of the latter isomorphisms, the lower row is the exact sequence of (3.42) with $(\mathcal{F}_1,\mathcal{F}_2)$ taken to be $(\tilde{\mathcal{F}}^{\mathfrak{n}},\tilde{\mathcal{F}}^{\mathfrak{m}})$ and directly gives the sequence (5.14) in the case $\Phi = \tilde{\mathcal{F}}$. In addition, since the commutativity of the second square implies that the second map in the upper row is $(\hat{v}_{\mathfrak{q}})_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})}$, the two rows combine to give the sequence (5.14) in the case $\Phi = \tilde{\mathscr{F}}$.
In a similar way, the long exact cohomology sequence of (5.11) combines with the descriptions of cohomology in (iii) and the appropriate case of (3.42) to give an exact commutative diagram

$$\begin{array}{ccccccccc}
H^1_{\tilde{\mathscr{F}}(\mathfrak{m})}(k,\mathcal{A}) & \hookrightarrow & H^1_{\tilde{\mathscr{F}}^{\mathfrak{m}}}(k,\mathcal{A}) & \longrightarrow & \bigoplus_{\mathfrak{q}\in V(\mathfrak{m})} H^1_{/\text{tr}}(k_{\mathfrak{q}},\mathcal{A}) & & & & \\
\downarrow & & \downarrow & & \cong \downarrow (\beta_{\mathfrak{q}})_{\mathfrak{q}} & & & & \\
H^1_{\tilde{\mathcal{F}}(\mathfrak{m})}(k,\mathcal{A}) & \hookrightarrow & H^1_{\tilde{\mathcal{F}}^{\mathfrak{m}}}(k,\mathcal{A}) & \xrightarrow{(\psi^{\text{fs}}_{\mathfrak{q}})_{\mathfrak{q}}} & \mathbb{A}^{\nu(\mathfrak{m})} & \longrightarrow & H^1_{\tilde{\mathcal{F}}^*(\mathfrak{m})}(k,\mathcal{B})^* & \twoheadrightarrow & H^1_{(\tilde{\mathcal{F}}^*)_{\mathfrak{m}}}(k,\mathcal{B})^*.
\end{array}$$

Here, for each $\mathfrak{q}$, we write $\beta_{\mathfrak{q}}$ for the isomorphism $H^1_{/\text{tr}}(k_{\mathfrak{q}},\mathcal{A}) \cong \mathbb{A}$ induced by the composite map $H^1(k_{\mathfrak{q}},\mathcal{A}) \to \mathcal{A}/(\tau-1) \cong \mathbb{A}$ that occurs in the definition of $\psi^{\text{fs}}_{\mathfrak{q}}$. It follows that the second map in the upper row is $(\hat{\psi}^{\text{fs}}_{\mathfrak{q}})_{\mathfrak{q}\in V(\mathfrak{m})}$ and so the exact sequence (5.15) with $\Phi = \tilde{\mathcal{F}}$, respectively $\Phi = \tilde{\mathscr{F}}$, follows directly from the lower row of the above diagram, respectively from a comparison of the upper and lower rows of the diagram.
The derivations of (5.16) and (5.17) from the respective exact triangles (5.12) and (5.13) follow along precisely similar lines. For brevity, we therefore leave details of these derivations to the reader except to note that, for each $\mathfrak{q} \in V(\mathfrak{n})$, the isomorphism (5.19) induces an isomorphism of $H^1_{\text{tr}}(k_{\mathfrak{q}},\mathcal{A}) \cong H^1_{/f}(k_{\mathfrak{q}},\mathcal{A})$ with $\mathbb{A}$ and, for each $\mathfrak{q} \in V(\mathfrak{a})$, the map $\psi^{\text{fs}}_{\mathfrak{q}}$ relies on the isomorphism of $H^1_f(k_{\mathfrak{q}},\mathcal{A}) = \mathcal{A}/(\tau-1)$ with $\mathbb{A}$ that was fixed earlier. □

The $\mathcal{R}$-module Y fixed at the beginning of §4.3 gives rise to an $\mathbb{A}$-module $\tilde{Y} := Y \otimes_{\mathcal{R}} \mathbb{A}$ that is non-zero, free and identifies, via (3.46), with a quotient of $X(\tilde{\mathscr{F}})$. In the next result we shall use variants of the exact triangles (5.10) and (5.11) that are respectively constructed via the following diagrams in $D(\mathbb{A})$

$$\begin{array}{ccccccc}
(\tilde{Y}\oplus M(\mathfrak{n}))[-1] & \longrightarrow & (\tilde{Y}\oplus M(\mathfrak{m}))[-1] & \longrightarrow & M(\mathfrak{m}/\mathfrak{n})[-1] & \longrightarrow & \cdot \\
\theta_{\mathfrak{n}}\uparrow & & \theta_{\mathfrak{m}}\uparrow & & \theta_{\mathfrak{m},\mathfrak{n}}\uparrow & & \\
C(\tilde{\mathscr{F}}^{\mathfrak{n}}) & \longrightarrow & C(\tilde{\mathscr{F}}^{\mathfrak{m}}) & \longrightarrow & \bigoplus_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})} C_{\mathfrak{q}} & \longrightarrow & \cdot \\
\uparrow & & \uparrow & & \uparrow & & \\
C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{n}}) & \xrightarrow{\rho_1} & C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}}) & \xrightarrow{\rho_1'} & \bigoplus_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})} H^1_{/f}(k_{\mathfrak{q}},\mathcal{A})[0] & \longrightarrow & \cdot
\end{array} \tag{5.20}$$

$$
\begin{array}{ccccccc}
\tilde{Y}[-1] & \longrightarrow & (\tilde{Y}\oplus M(\mathfrak{m}))[-1] & \longrightarrow & M(\mathfrak{m})[-1] & \longrightarrow & \cdot \\
\uparrow{\scriptstyle \theta'_{\mathfrak{m}}} & & \uparrow{\scriptstyle \theta_{\mathfrak{m}}} & & \uparrow{\scriptstyle \theta''_{\mathfrak{m}}} & & \\
C(\tilde{\mathscr{F}}(\mathfrak{m})) & \longrightarrow & C(\tilde{\mathscr{F}}^{\mathfrak{m}}) & \longrightarrow & \bigoplus_{\mathfrak{q}\in V(\mathfrak{m})} C'_{\mathfrak{q}} & \longrightarrow & \cdot \\
\uparrow & & \uparrow & & \| & & \\
C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m})) & \xrightarrow{\rho_2} & C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}}) & \xrightarrow{\rho'_2} & \bigoplus_{\mathfrak{q}\in V(\mathfrak{m})} H^1_{/\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{A})[0] & \longrightarrow & \cdot
\end{array}
\tag{5.21}
$$

Here, for each modulus $\mathfrak{a}$ in $\mathcal{N}$, we write $M(\mathfrak{a})$ for the free $\mathbb{A}$-module $\bigoplus_{\mathfrak{q}\in V(\mathfrak{a})} H^0(k_{\mathfrak{q}},\mathcal{B})^*$ so that the upper rows of the respective diagrams are the exact triangles induced by the obvious short exact sequence $0\to M(\mathfrak{n})\to M(\mathfrak{m})\to M(\mathfrak{m}/\mathfrak{n})\to 0$ and $0\to \tilde{Y}\to \tilde{Y}\oplus M(\mathfrak{m})\to M(\mathfrak{m})\to 0$. In addition, we set

$$C_{\mathfrak{q}} := H^1_{/f}(k_{\mathfrak{q}},\mathcal{A})[0]\oplus H^2(k_{\mathfrak{q}},\mathcal{A})[-1], \quad \text{respectively } C'_{\mathfrak{q}} := H^1_{/\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{A})[0]\oplus H^2(k_{\mathfrak{q}},\mathcal{A})[-1],$$

for $\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})$, respectively $\mathfrak{q}\in V(\mathfrak{m})$, so that the central rows of the two diagrams are respectively the exact triangles of (5.10) and (5.11). The morphisms $\theta'_{\mathfrak{m}}$ and $\theta_{\mathfrak{m}}$ are the unique morphisms in $D(\mathbb{A})$ for which $H^1(\theta'_{\mathfrak{m}})$ and $H^1(\theta_{\mathfrak{m}})$ are the surjective maps induced by the descriptions of $H^1(C(\tilde{\mathscr{F}}(\mathfrak{m})))$ and $H^1(C(\tilde{\mathscr{F}}^{\mathfrak{m}}))$ in Proposition 5.9 (ii) (these morphisms exist and are unique since $C(\tilde{\mathscr{F}}(\mathfrak{m}))$ and $C(\tilde{\mathscr{F}}^{\mathfrak{m}})$ are both acyclic in degrees greater than one) and we write $C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))$ and $C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})$ for their respective mapping fibres. Finally, we write $\theta_{\mathfrak{m},\mathfrak{n}}$ and $\theta''_{\mathfrak{m}}$ for the unique morphisms in $D(\mathbb{A})$ for which $H^1(\theta_{\mathfrak{m},\mathfrak{n}})$ and $H^1(\theta''_{\mathfrak{m}})$ are the direct sums over $V(\mathfrak{m}/\mathfrak{n})$ and $V(\mathfrak{m})$ of the local duality isomorphisms $H^2(k_{\mathfrak{q}},\mathcal{A})\cong H^0(k_{\mathfrak{q}},\mathcal{B})^*$ so that the respective mapping fibres identify with $\bigoplus_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})} H^1_{/f}(k_{\mathfrak{q}},\mathcal{A})[0]$ and $\bigoplus_{\mathfrak{q}\in V(\mathfrak{m})} H^1_{/\mathrm{tr}}(k_{\mathfrak{q}},\mathcal{A})[0]$ and the right hand columns of the diagrams are the associated exact triangles. At this point, we note that the upper squares of both diagrams commute and so the Octahedral axiom implies the existence of the indicated morphisms ρ_1,ρ'_1,ρ_2 and ρ'_2 that make the respective lower rows exact triangle and the whole diagrams commutative (in $D(\mathbb{A})$).

(5.22) Proposition. *Recall the integer $r = r_{\mathfrak{F},Y}$ defined in (4.18). Then for all moduli $\mathfrak{m}$ and $\mathfrak{n}$ in $\mathcal{N}$ with $\mathfrak{n}\mid\mathfrak{m}$, the following claims are valid.*

(i) $C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))$ and $C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})$ belong to $D^{\mathrm{perf}}(\mathbb{A})$ and are such that, in $K_0(\mathbb{A})$, one has

$$\chi_{\mathbb{A}}(C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))) = r\cdot[\mathbb{A}] \quad \textit{and} \quad \chi_{\mathbb{A}}(C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})) = (r+\nu(\mathfrak{m}))\cdot[\mathbb{A}].$$

(ii) $C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))$ and $C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})$ are acyclic outside degrees zero and one. There are identifications $H^0(C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m})) = H^1_{\tilde{\mathscr{F}}(\mathfrak{m})}(k,\mathcal{A})$ and $H^0(C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})) = H^1_{\tilde{\mathscr{F}}^{\mathfrak{m}}}(k,\mathcal{A})$ and canonical short exact sequences

$$0\to H^1_{\tilde{\mathcal{F}}^*(\mathfrak{m})}(k,\mathcal{B})^*\to H^1(C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m})))\to \ker(X(\tilde{\mathscr{F}})\to\tilde{Y})\to 0$$

$$0\to H^1_{(\tilde{\mathcal{F}}^*)_{\mathfrak{m}}}(k,\mathcal{B})^*\to H^1(C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}}))\to \ker(X(\tilde{\mathscr{F}})\to\tilde{Y})\to 0.$$

(iii) If $r>0$, then there exists a commutative diagram of $\mathbb{A}$-modules

$$
\begin{array}{ccc}
\mathrm{Det}_{\mathbb{A}}(C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{n}})) & \xrightarrow{\vartheta_{\tilde{\mathscr{F}}^{\mathfrak{n}},\tilde{Y}}} & \bigcap_{\mathbb{A}}^{r+\nu(\mathfrak{n})} H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A}) \\
\uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \bigwedge_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})}\hat{v}_{\mathfrak{q}}} \\
\mathrm{Det}_{\mathbb{A}}(C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})) & \xrightarrow{\vartheta_{\tilde{\mathscr{F}}^{\mathfrak{m}},\tilde{Y}}} & \bigcap_{\mathbb{A}}^{r+\nu(\mathfrak{m})} H^1_{\tilde{\mathscr{F}}^{\mathfrak{m}}}(k,\mathcal{A}) \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \bigwedge_{\mathfrak{q}\in V(\mathfrak{m})}\hat{\psi}^{\mathrm{fs}}_{\mathfrak{q}}} \\
\mathrm{Det}_{\mathbb{A}}(C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))) & \xrightarrow{\vartheta_{\tilde{\mathscr{F}}(\mathfrak{m}),\tilde{Y}}} & \bigcap_{\mathbb{A}}^{r} H^1_{\tilde{\mathscr{F}}(\mathfrak{m})}(k,\mathcal{A}).
\end{array}
$$

The maps $\vartheta_{\tilde{\mathscr{F}}^{\mathfrak{m}},\tilde{Y}}$ and $\vartheta_{\tilde{\mathscr{F}}(\mathfrak{m}),\tilde{Y}}$ are defined in the course of the proof below and have cokernels that are respectively annihilated by $\mathrm{Fitt}^0_{\mathbb{A}}(H^1(C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})))$ and $\mathrm{Fitt}^0_{\mathbb{A}}(H^1(C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))))$.

The maps φ_1 and φ_2 are isomorphisms and arise as follows: φ_1 is induced by the lower row of (5.20) and the identification $\mathrm{Det}_{\mathbb{\Lambda}}(H^1_{/f}(k_\mathfrak{q},\mathcal{A})[0]) \cong \mathrm{Det}_{\mathbb{\Lambda}}(\mathbb{\Lambda}[0]) = \mathbb{\Lambda}$ for each $\mathfrak{q} \in V(\mathfrak{m}/\mathfrak{n})$ given by the isomorphism $H^1_{/f}(k_\mathfrak{q},\mathcal{A}) \cong \mathbb{\Lambda}$ induced by (5.19); φ_2 is induced by the lower row of (5.21) and the identification $\mathrm{Det}_{\mathbb{\Lambda}}(H^1_{/\mathrm{tr}}(k_\mathfrak{q},\mathcal{A})[0]) \cong \mathrm{Det}_{\mathbb{\Lambda}}(\mathbb{\Lambda}[0]) = \mathbb{\Lambda}$ for each $\mathfrak{q} \in V(\mathfrak{m})$ given by the isomorphism $H^1_{/\mathrm{tr}}(k_\mathfrak{q},\mathcal{A}) \cong \mathbb{\Lambda}$ induced by (5.5).

Proof. The $\mathbb{\Lambda}$-module $M(\mathfrak{m})$ is free of rank $\nu(\mathfrak{m})$. Hence, by combining the relevant cases of Proposition 5.9 (i) with the exact triangles given by the first two columns of (5.21), one deduces $C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))$ and $C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})$ belong to $D^{\mathrm{perf}}(\mathbb{\Lambda})$, and also that, in $K_0(\mathbb{\Lambda})$, one has both

$$\begin{aligned}\chi_{\mathbb{\Lambda}}\big(C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))\big) &= \chi_{\mathbb{\Lambda}}\big(C(\tilde{\mathscr{F}}(\mathfrak{m}))\big) - \chi_{\mathbb{\Lambda}}(\tilde{Y}[-1]) \\ &= \chi_{\mathbb{\Lambda}}\big(C(\tilde{\mathscr{F}})\big) + [\tilde{Y}] \\ &= r\cdot[\mathbb{\Lambda}]\end{aligned}$$

and

$$\begin{aligned}\chi_{\mathbb{\Lambda}}\big(C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})\big) &= \chi_{\mathbb{\Lambda}}\big(C(\tilde{\mathscr{F}}^{\mathfrak{m}})\big) - \chi_{\mathbb{\Lambda}}(M(\tilde{Y},\mathfrak{m})[-1]) \\ &= \chi_{\mathbb{\Lambda}}\big(C(\tilde{\mathscr{F}})\big) + [\tilde{Y}] + \nu(\mathfrak{m})\cdot[\mathbb{\Lambda}] \\ &= (r+\nu(\mathfrak{m}))\cdot[\mathbb{\Lambda}].\end{aligned}$$

This proves (i). In addition, all assertions in (ii) are obtained by combining the relevant cases of Proposition 5.9 (iii) with an analysis of the long exact cohomology sequences of the first two columns of (5.21).

We note now that the results of (i) and (ii) combine with the argument of Lemma 2.35 to imply the hypotheses of Proposition A.9(iv) are satisfied by both of the following sets of data

- $C^\bullet, D^\bullet, r$ and n are respectively taken to be $C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})$, $C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{n}})$, $r+\nu(\mathfrak{n})$ and $\nu(\mathfrak{m}/\mathfrak{n})$ and the exact triangle (A.10) is the lower row of (5.20);
- $C^\bullet, D^\bullet, r$ and n are respectively taken to be $C_{\tilde{Y}}(\tilde{\mathscr{F}}(\mathfrak{m}))$, $C_{\tilde{Y}}(\tilde{\mathscr{F}}^{\mathfrak{m}})$, r and $\nu(\mathfrak{m})$ and the exact triangle (A.10) is the lower row of (5.21).

By applying Lemma 2.35 to the first, respectively second, set of data we obtain the upper, respectively, lower square in the commutative diagram of (iii), with the assertions concerning annihilation of cokernels following directly from Proposition A.9 (i). Here we also use the fact that the argument of Proposition 5.9 (v) explicitly describes the map $H^0(\rho_1')$ induced by the lower row of (5.20) in terms of the maps $\hat{v}_\mathfrak{q}$ for $\mathfrak{q}$ in $V(\mathfrak{m}/\mathfrak{n})$ and the map $H^0(\rho_2')$ induced by the lower row of (5.21) in terms of the maps $\hat{\psi}^{\mathrm{fs}}_\mathfrak{q}$ for $\mathfrak{q}$ in $V(\mathfrak{m})$. □

5.3. The Kolyvagin derivative homomorphism

In this subsection, we fix a family of Nekovář structures $\mathfrak{F} = (\mathscr{F}_K)_{K\in\Omega}$ satisfying Hypothesis 4.1 and natural numbers t and i and continue to use the notation in (5.1) (so that $\tilde{\mathscr{F}}$ denotes $\mathscr{F}_{k,i}$ etc.). We shall first define a notion of Kolyvagin system of rank t for the Nekovář structure $\tilde{\mathscr{F}}$ and then refine arguments from [26] in order to prove that Euler systems in $\mathrm{ES}^t(\mathfrak{F})$ give rise to such Kolyvagin systems via a natural 'derivative homomorphism' construction.

5.3.1. Kolyvagin systems for Nekovář structures

Fix a modulus $\mathfrak{n} \in \mathcal{N}$ and a prime $\mathfrak{q} \in \mathcal{Q} \setminus V(\mathfrak{n})$. Then, by applying Lemma 2.17 (ii) to the exact sequence (5.16) (with $\mathfrak{m}$ and $\mathfrak{n}$ replaced by $\mathfrak{n}$ and $\mathfrak{q}$), one obtains a map of $\mathbb{\Lambda}$-modules

$$\check{v}_\mathfrak{q}\colon \bigcap\nolimits^t_{\mathbb{\Lambda}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A}) \to \bigcap\nolimits^{t-1}_{\mathbb{\Lambda}} H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k,\mathcal{A}).$$

In the same way, by applying Lemma 2.17 (ii) to (5.17) (with $\mathfrak{a}$ and $\mathfrak{m}$ replaced by $\mathfrak{q}$ and $\mathfrak{n}$), one obtains a map of Λ-modules

$$\check{\psi}^{\mathrm{fs}}_\mathfrak{q}\colon \bigcap\nolimits^t_{\mathbb{\Lambda}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A}) \to \bigcap\nolimits^{t-1}_{\mathbb{\Lambda}} H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k,\mathcal{A}).$$

We can now give a definition of Kolyvagin system that is appropriate for our theory.

(5.23) Definition. *A 'Kolyvagin system' of rank t for the Nekovář structure $\tilde{\mathscr{F}}$ is a family*

$$(\kappa_{\mathfrak{n}})_{\mathfrak{n}\in\mathcal{N}} \in \prod\nolimits_{\mathfrak{n}\in\mathcal{N}} \bigcap\nolimits_{\mathbb{A}}^{t} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})$$

with the property that, for every $\mathfrak{n}\in\mathcal{N}$ and $\mathfrak{q}\in\mathcal{Q}\setminus V(\mathfrak{n})$, the 'finite-singular relation'

$$\check{v}_{\mathfrak{q}}(\kappa_{\mathfrak{nq}}) = \check{\psi}^{\rm fs}_{\mathfrak{q}}(\kappa_{\mathfrak{n}})$$

is valid in $\bigcap_{\mathbb{A}}^{t-1} H^1_{\tilde{\mathscr{F}}_{\mathfrak{q}}(\mathfrak{n})}(k,\mathcal{A})$. The collection of all such families is naturally a $\mathbb{A}$-module and we denote this by $\mathrm{KS}^t(\tilde{\mathscr{F}})$.

To prepare for the statement of our main result concerning these systems, we note that, for each $K\in\Omega$, Hypothesis 4.1 implies the Nekovář structure $\mathscr{F}_K$ satisfies Hypotheses 3.44 (with $R=\mathcal{R}[\mathcal{G}_K]$). It follows that Lemma 2.37 (ii) combines with Proposition 5.9 (iii) and (v) to imply the existence of a natural 'projection' map of $\mathcal{R}[\mathcal{G}_K]$-modules

$$\pi^t_{K,\tilde{\mathscr{F}}}\colon \bigcap\nolimits^t_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathfrak{F}}(K,\mathcal{T}) \to \bigcap\nolimits^t_{\mathbb{A}[\mathcal{G}_K]} H^1_{\tilde{\mathscr{F}}}(K,\mathcal{A}). \tag{5.24}$$

For each modulus $\mathfrak{n}\in\mathcal{N}$, we set

$$G_{\mathfrak{n}} \coloneqq \mathrm{Gal}(k(\mathfrak{n})/k(1)) \cong \prod\nolimits_{\mathfrak{q}\in V(\mathfrak{n})} G_{\mathfrak{q}},$$

and we fix a pre-image $\mathrm{N}_{\mathfrak{n}}$ of $\mathrm{N}_{\mathcal{G}_{k(1)}}$ under the (surjective) projection map $\mathbb{Z}[\mathcal{G}_{k(\mathfrak{n})}]\to\mathbb{Z}[\mathcal{G}_{k(1)}]$. By using the generator $\sigma_{\mathfrak{q}}$ of $G_{\mathfrak{q}}$ for $\mathfrak{q}\in\mathcal{Q}_i$ fixed at the beginning of § 5, we then define 'derivative operators'

$$D_{\mathfrak{q}} \coloneqq \sum\nolimits_{j\in[|G_{\mathfrak{q}}|-1]} j\sigma_{\mathfrak{q}}^j \in \mathbb{Z}[G_{\mathfrak{q}}], \quad D_{\mathfrak{n}} \coloneqq \prod\nolimits_{\mathfrak{q}\in V(\mathfrak{n})} D_{\mathfrak{q}} \in \mathbb{Z}[G_{\mathfrak{n}}] \quad \text{and} \quad D'_{\mathfrak{n}} \coloneqq D_{\mathfrak{n}}\cdot \mathrm{N}_{\mathfrak{n}} \in \mathbb{Z}[\mathcal{G}_{k(\mathfrak{n})}].$$

The following result is well-known.

(5.25) Lemma. *Fix $c=(c_K)_{K\in\Omega}$ of $\mathrm{ES}^t(\mathfrak{F})$. Then, for every modulus $\mathfrak{n}\in\mathcal{N}$, the element*

$$D'_{\mathfrak{n}}\cdot \pi^t_{k(\mathfrak{n}),\tilde{\mathscr{F}}}(c_{k(\mathfrak{n})}) \in \bigcap\nolimits^t_{\mathbb{A}[\mathcal{G}_{k(\mathfrak{n})}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}),\mathcal{A})$$

is both fixed by $\mathcal{G}_{k(\mathfrak{n})}$ and independent of the choice of lift $\mathrm{N}_{\mathfrak{n}}$ of $\mathrm{N}_{\mathcal{G}_{k(1)}}$.

Proof. Both claims are true if, for all $\mathfrak{n}$, one has $(\sigma-1)D_{\mathfrak{n}}(\pi^t_{K,\tilde{\mathscr{F}}}(c_{k(\mathfrak{n})}))=0$ for every $\sigma\in G_{\mathfrak{n}}$. Hence, since $\mathfrak{a}_i$ annihilates $\bigcap^t_{\mathbb{A}[\mathcal{G}_{k(\mathfrak{n})}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}),\mathcal{A})$, it is enough to show that every element $(\sigma-1)D_{\mathfrak{n}}(c_{k(\mathfrak{n})})$ belongs to $\mathfrak{a}_i\cdot\bigcap^t_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}}(k(\mathfrak{n}),\mathcal{T})$. This is proved by the argument of [26, Lem. 6.12], which we now briefly explain for the convenience of the reader.

We use induction on $\nu(\mathfrak{n})$ and may assume both that $\nu(\mathfrak{n})>1$ (since there is nothing to prove if $\mathfrak{n}=1$) and also $\sigma=\sigma_{\mathfrak{q}}$ for some $\mathfrak{q}\in V(\mathfrak{n})$ (since $G_{\mathfrak{n}}$ is generated by the set $\{\sigma_{\mathfrak{q}}:\mathfrak{q}\in V(\mathfrak{n})\}$). Then, since $(\sigma_{\mathfrak{q}}-1)D_{\mathfrak{q}}=|G_{\mathfrak{q}}|-\mathrm{N}_{G_{\mathfrak{q}}}$, one has

$$\begin{aligned}(\sigma_{\mathfrak{q}}-1)D'_{\mathfrak{n}}c_{k(\mathfrak{n})} &= (|G_{\mathfrak{q}}|-\mathrm{N}_{G_{\mathfrak{q}}})D'_{\mathfrak{n}/\mathfrak{q}}c_{k(\mathfrak{n})}\\ &= |G_{\mathfrak{q}}|D'_{\mathfrak{n}/\mathfrak{q}}c_{k(\mathfrak{n})} - \nu^t_{k(\mathfrak{nq})/k(\mathfrak{n})}(D'_{\mathfrak{n}/\mathfrak{q}}\cdot \mathrm{Eul}_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}}^{-1})c_{k(\mathfrak{n}/\mathfrak{q})}),\end{aligned}$$

where the second equality holds by the Euler system norm relations. Now, since $\mathfrak{q}\in\mathcal{Q}$, the element $\mathrm{Eul}_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}}^{-1})$ belongs to the augmentation ideal of $\mathbb{A}[G_{\mathfrak{n}/\mathfrak{q}}]$ and so the induction hypothesis implies $D'_{\mathfrak{n}/\mathfrak{q}}\cdot\mathrm{Eul}_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}}^{-1})c_{k(\mathfrak{n}/\mathfrak{q})}$ belongs to $\mathfrak{a}_i\cdot\bigcap^t_{\mathbb{A}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}/\mathfrak{q}),\mathcal{A})$. In addition, the containment $\mathfrak{q}\in\mathcal{Q}$ also implies that $|\mathbb{A}|$ divides $|G_{\mathfrak{q}}|$. Since $|\mathbb{A}|$ belongs to $\mathfrak{a}_i$, this shows that $(\sigma_{\mathfrak{q}}-1)D'_{\mathfrak{n}}c_{k(\mathfrak{n})}$ is contained in $\mathfrak{a}_i\cdot\bigcap^t_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}}(k(\mathfrak{n}),\mathcal{T})$, as required. □

For each modulus $\mathfrak{n}\in\mathcal{N}$ the Nekovář structure $\tilde{\mathscr{F}}^{k(\mathfrak{n})}$ agrees with $\tilde{\mathscr{F}}^{\mathfrak{n}}$ (cf. Remark 5.8). As a consequence, the assumed validity of Hypothesis 4.1 combines with the argument of Lemma 4.5 to imply the existence of a canonical isomorphism of $\mathbb{A}$-modules

$$\nu^t_{k(\mathfrak{n})/k,\tilde{\mathscr{F}}}\colon \bigcap\nolimits^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A}) \xrightarrow{\simeq} \big(\bigcap\nolimits^t_{\mathbb{A}[\mathcal{G}_{k(\mathfrak{n})}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}),\mathcal{A})\big)^{\mathcal{G}_{k(\mathfrak{n})}}.$$

For each $\mathfrak{l} \in \mathcal{Q} \setminus V(\mathfrak{n})$, we may regard $\mathrm{Eul}_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}^{-1})$ as an element of $\mathcal{R}[G_{\mathfrak{n}}]$. Then, writing $I_{\mathfrak{n}}$ for the augmentation ideal of $\mathbb{Z}[G_{\mathfrak{n}}]$, the definition of $\mathcal{Q}$ implies the image of $\mathrm{Eul}_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}^{-1})$ in $\mathbb{A}[G_{\mathfrak{n}}]$ belongs to $\mathbb{A} \otimes_{\mathbb{Z}} I_{\mathfrak{n}}$ and so defines an element of $\mathbb{A} \otimes_{\mathbb{Z}} (I_{\mathfrak{n}}/I_{\mathfrak{n}}^2)$. In particular, for every $\mathfrak{q} \neq \mathfrak{l}$, we may define an element $x_{\mathfrak{l}}^{(\mathfrak{q})}$ of $\mathbb{A}$ by means of the equality

$$\mathrm{Eul}_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}^{-1}) = x_{\mathfrak{l}}^{(\mathfrak{q})} \otimes (\sigma_{\mathfrak{q}} - 1) \quad \text{in} \quad \mathbb{A} \otimes_{\mathbb{Z}} (I_{\mathfrak{q}}/I_{\mathfrak{q}}^2).$$

We also write $\mathfrak{S}(\mathfrak{n})$ for the set $\mathrm{Per}(V(\mathfrak{n}))$ of permutations of the set $V(\mathfrak{n})$.

(5.26) Definition. *Fix an element $c = (c_K)_{K\in\Omega}$ of $\mathrm{ES}^t(\mathfrak{F})$. Then, for each modulus $\mathfrak{n} \in \mathcal{N}$, Lemma 5.25 allows us to define*

$$\kappa'(c)_{\mathfrak{n}} := (\nu^t_{k(\mathfrak{n})/k,\tilde{\mathscr{F}}})^{-1}(\pi^t_{k(\mathfrak{n}),\tilde{\mathscr{F}}}(D'_{\mathfrak{n}}(c_{k(\mathfrak{n})}))) \quad \in {\bigcap}^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A}).$$

We then set

$$\kappa(c)_{\mathfrak{n}} := \sum\nolimits_{\tau\in\mathfrak{S}(\mathfrak{n})} \mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{q}\in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{q})}_{\tau(\mathfrak{q})}\big) \cdot \kappa'(c)_{\mathfrak{d}_\tau} \quad \in {\bigcap}^a_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A}),$$

where $\mathfrak{d}_\tau$ denotes the product over all $\mathfrak{q} \in V(\mathfrak{n})$ with $\tau(\mathfrak{q}) = \mathfrak{q}$, and define the 'Kolyvagin derivative' of c to be the family

$$\kappa(c) := (\kappa(c)_{\mathfrak{n}})_{\mathfrak{n}\in\mathcal{N}} \in \prod\nolimits_{\mathfrak{n}\in\mathcal{N}} {\bigcap}^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A}).$$

We can now finally state the main result of § 5.3.

(5.27) Theorem. *Assume the family $\mathfrak{F} = (\mathscr{F}_K)_{K\in\Omega}$ of Nekovář structures satisfies Hypothesis 4.1 and that $\mathcal{K}$ and $\mathcal{T}$ satisfy Hypothesis 4.16. Fix an element c of $\mathrm{ES}^t(\mathfrak{F})$. Then, for each $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{q} \in V(\mathfrak{n})$, one has*

$$\kappa(c)_{\mathfrak{n}} \in {\bigcap}^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})$$

and

$$\check{v}_{\mathfrak{q}}(\kappa(c)_{\mathfrak{n}}) = \check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}(\kappa(c)_{\mathfrak{n}/\mathfrak{q}}) \in {\bigcap}^{t-1}_{\Lambda} H^1_{\tilde{\mathscr{F}}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{q})}(k,\mathcal{A}).$$

In particular, the assignment $c \mapsto \kappa(c)$ defines a well-defined homomorphism of $\mathcal{R}$-modules

$$\mathrm{ES}^t(\mathfrak{F}) \to \mathrm{KS}^t(\tilde{\mathscr{F}}).$$

After some preliminary steps, the proof of this result will be completed in § 5.3.3.

5.3.2. The reduction of Theorem 5.27 to rank one

If $\mathfrak{F}$ is the family $\mathfrak{F}_{\mathrm{rel}}(\mathcal{T})$ of relaxed Nekovář structures discussed in Example 4.9, $\mathcal{R} = \mathcal{O}[\mathcal{G}_K]$ for a finite extension $\mathcal{O}$ of $\mathbb{Z}_p$ and a field $K \in \mathcal{K}$ and the filtration $(\mathfrak{a}_n)_n$ is $(p^n\mathcal{R})_n$, then the result of Theorem 5.27 coincides with [26, Th. 6.15]. We recall that the latter result is proved by first adapting a technique of Rubin [91] and Perrin-Riou [87] to reduce to the case $a = 1$, and then deriving this special case from an argument used by Mazur and Rubin to prove [74, Th. 3.2.4]. We shall use the same strategy to prove Theorem 5.27 for general Nekovář structures, though in the case $t = 1$ we also now rely on a suitably modified version of arguments of Kato in [59, § 2].

As a first step, we must therefore establish a version of the key technical result [26, Lem. 6.22] that is appropriate for Nekovář structures.

(5.28) Proposition. *Fix a modulus $\mathfrak{n}$ in $\mathcal{N}$. Then, for each $\Phi \in {\bigwedge}^{t-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A})^*$, there exists a family of elements*

$$\Psi = (\Psi_K)_{K\in\Omega} \in \prod\nolimits_{K\in\Omega} {\bigwedge}^{t-1}_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}}(K,\mathcal{T})^*$$

with both of the following properties.

(i) *For all K and L in Ω with $K \subseteq L$, and all $y \in \bigcap^t_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}}(K,\mathcal{T})$, one has*

$$(\Psi_L \circ \nu^t_{L/K})(y) = (\nu^1_{L/K} \circ \Psi_K)(y)$$

in $H^1_{\mathscr{F}}(L,\mathcal{T})$, where the maps $\nu^t_{L/K}$ and $\nu^1_{L/K}$ are as defined in Lemma 4.5.

(ii) *For each divisor $\mathfrak{m}$ of $\mathfrak{n}$, we use the first map in (5.14) to regard $H^1_{\tilde{\mathscr{F}}^{\mathfrak{m}}}(k,\mathcal{A})$ as a submodule of $H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A})$. Then, in $H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A})$, one has*

$$\Phi((\nu^t_{k(\mathfrak{m})/k,\tilde{\mathscr{F}}})^{-1}(D'_{\mathfrak{m}} \cdot \pi^t_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(c_{k(\mathfrak{m})}))) = (\nu^1_{k(\mathfrak{m})/k,\tilde{\mathscr{F}}})^{-1}(D'_{\mathfrak{m}} \cdot \pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(\Psi_{k(\mathfrak{m})}(c_{k(\mathfrak{m})}))).$$

Proof. For any K and L in Ω with $K \subseteq L$, the canonical injective map (4.4) induces a restriction map $\bigwedge^{t-1}_{\mathcal{R}[\mathcal{G}_L]} H^1_{\mathscr{F}}(L,\mathcal{T})^* \to \bigwedge^{t-1}_{\mathcal{R}[\mathcal{G}_K]} H^1_{\mathscr{F}}(K,\mathcal{T})^*$. Via these maps, the construction of Ψ is reduced to the consideration of a cofinal subset of Ω.

We therefore fix a cofinal subset $\{F_n\}_{n\in\mathbb{N}}$ of Ω that is totally ordered with respect to inclusion, order-isomorphic to $\mathbb{N}$ and such that $F_1 = k(\mathfrak{n})$ and, for each n, we set

$$\mathcal{S}_n := \mathcal{R}[\mathcal{G}_{F_n}] \quad \text{and} \quad C_n := C(\mathscr{F}_{F_n}).$$

Then, to construct a family of the required sort, it is enough to inductively construct elements Ψ_{F_n} of $\bigwedge^{t-1}_{\mathcal{S}_n} H^1_{\mathscr{F}}(F_n,\mathcal{T})^*$ that have the claimed properties. To do this, we first inductively construct a family of finitely generated free $\mathcal{S}_n$-modules P_n such that C_n has a resolution $P_n \xrightarrow{d_n} P_n$ (with the first term placed in degree zero) and a family of elements $f_n \in \bigwedge^{t-1}_{\mathcal{S}_n} P_n^*$ with certain compatibility properties.

To construct the required objects for $n = 1$, we note Lemma 2.31 (i) identifies $H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A})$ with the $G_{\mathfrak{n}}$-invariants of $H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}),\mathcal{A})$. Hence, upon taking duals over the self-injective ring $\mathbb{A}[G_{\mathfrak{n}}]$, it follows that the restriction map

$$\bigwedge^{t-1}_{\mathbb{A}[G_{\mathfrak{n}}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}),\mathcal{A})^* \to \bigwedge^{t-1}_{\mathbb{A}[G_{\mathfrak{n}}]} \big(H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}),\mathcal{A})^{G_{\mathfrak{n}}}\big)^* \cong \bigwedge^{t-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}^{\mathfrak{n}}}(k,\mathcal{A})^* \tag{5.29}$$

is surjective and so we can fix a pre-image $\widetilde{\Phi}$ of Φ under this map. Now, since $\mathscr{F}_{F_1}$ validates Hypothesis 3.44, Lemma 2.35 implies C_1 has a resolution $P_1 \xrightarrow{d_1} P_1$, where P_1 is a finitely generated free $\mathcal{S}_1$-module (and the first term is placed in degree zero). Hence, from Proposition 3.45 (iii), it follows that $C(\mathscr{F}_{F_1} \otimes_{\mathcal{R}} \mathbb{A}) \cong C_1 \otimes^{\mathbb{L}}_{\mathcal{R}} \mathbb{A}$ is isomorphic to $P_{1,i} \xrightarrow{d_{1,i}} P_{1,i}$ with $P_{1,i} := P_1 \otimes_{\mathcal{R}} \mathbb{A}$ and $d_{1,i} := d_1 \otimes_{\mathcal{R}} 1_{\mathbb{A}}$. In particular, since the natural composite map

$$\bigwedge^{t-1}_{\mathcal{S}_1} P_1^* \to \big(\bigwedge^{t-1}_{\mathcal{S}_1} P_1^*\big) \otimes_{\mathcal{R}} \mathbb{A} \cong \bigwedge^{t-1}_{\mathbb{A}[G_{\mathfrak{n}}]} P_{1,i}^* \to \bigwedge^{t-1}_{\mathbb{A}[G_{\mathfrak{n}}]} \ker(d_{1,i})^* \cong \bigwedge^{t-1}_{\mathbb{A}[G_{\mathfrak{n}}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}),\mathcal{A})^*$$

is surjective (where the isomorphism follows from the fact P_1 is $\mathcal{S}_1$-free, and the third map is surjective since $\Lambda[G_{\mathfrak{n}}]$ is self-injective), we can fix a pre-image f_1 of $\widetilde{\Phi}$ under this map.

Having constructed P_1 and f_1, we now pass to the inductive step. For this, we assume to be given a resolution $P_n \xrightarrow{d_n} P_n$ of C_n (in which P_n is a finitely generated free $\mathcal{S}_n$-module) and an element f_n of $\bigwedge^{t-1}_{\mathcal{S}_n} P_n^*$ and use them to construct similar objects for $n+1$ in place of n.

To do this, we set $\tilde{C}_n := C(\mathscr{F}_{F_{n+1}} \otimes_{\mathcal{S}_{n+1}} \mathcal{S}_n)$ and write Π_n for the set of places of F_n above those in $S(F_{n+1}) \setminus S(F_n)$. Then the relevant cases of Hypothesis 4.1 (ii) and Proposition 3.45 (vi) combine to imply the existence of a canonical exact triangle in $D^{\mathrm{perf}}(\mathcal{S}_n)$

$$C_n \xrightarrow{\alpha_n} \tilde{C}_n \to \bigoplus_{\mathfrak{q}\in\Pi_n} \mathrm{R}\Gamma_f(F_{n,\mathfrak{q}},\mathcal{T}^*(1))^*[-1] \to \cdot. \tag{5.30}$$

In addition, for $\mathfrak{q} \in \Pi_n$, the complex $\mathrm{R}\Gamma_f(F_{n,\mathfrak{q}},\mathcal{T}^*(1))^*$ has a resolution $Q(\mathfrak{q}) \to Q(\mathfrak{q})$, with $Q(\mathfrak{q})$ a finitely generated free $\mathcal{S}_n$-module (and the first term placed in degree zero). Hence, setting $Q_n := \bigoplus_{\mathfrak{q}\in\Pi_n} Q(\mathfrak{q})$, a standard mapping cone construction combines with the fixed resolution of C_n to imply $\tilde{C}_n$ is isomorphic in $D^{\mathrm{perf}}(\mathcal{S}_n)$ to a complex $P_n \oplus Q_n \xrightarrow{d'_n} P_n \oplus Q_n$ (with the first term placed in degree zero) in such a way that α_n is induced by a commutative diagram of $\mathcal{S}_n$-modules

$$\begin{array}{ccc} P_n & \xrightarrow{d_n} & P_n \\ \Big\downarrow{\scriptstyle \alpha_n^0} & & \Big\downarrow{\scriptstyle \alpha_n^1} \\ P_n \oplus Q_n & \xrightarrow{d'_n} & P_n \oplus Q_n, \end{array}$$

in which α_n^0 and α_n^1 are the natural inclusion maps. In particular, since restriction through α_n^0 induces a surjective map $\kappa : \bigwedge_{\mathcal{S}_n}^{t-1}(P_n \oplus Q_n)^* \to \bigwedge_{\mathcal{S}_n}^{t-1} P_n^*$, we can fix an element $\widetilde{f}_n$ of $\bigwedge_{\mathcal{S}_n}^{t-1}(P_n \oplus Q_n)^*$ with $\kappa(\widetilde{f}_n) = f_n$.

Next we note that Proposition 3.45 (iv) induces an isomorphism in $D^{\mathrm{perf}}(\mathcal{S}_n)$

$$C_{n+1} \otimes^{\mathbb{L}}_{\mathcal{S}_{n+1}} \mathcal{S}_n \cong \tilde{C}_n, \tag{5.31}$$

and hence also an isomorphism of $\mathcal{S}_n$-modules $H^1(C_{n+1}) \otimes_{\mathcal{S}_{n+1}} \mathcal{S}_n \cong H^1(\tilde{C}_n)$.

We now fix a finitely generated free $\mathcal{S}_{n+1}$-module P_{n+1} for which there exists an isomorphism of $\mathcal{S}_n$-modules from $(P_{n+1})_n := P_{n+1} \otimes_{\mathcal{S}_{n+1}} \mathcal{S}_n$ to $P_n \oplus Q_n$, and write ι_n for the induced surjective map of S_{n+1}-modules $P_{n+1} \twoheadrightarrow P_n \oplus Q_n$. The resolution of $\tilde{C}_n$ fixed above induces a surjective map of $\mathcal{S}_n$-modules $j_n \colon P_n \oplus Q_n \twoheadrightarrow H^1(\tilde{C}_n)$. Then, since P_{n+1} is a free $\mathcal{S}_{n+1}$-module, we can fix a commutative diagram of $\mathcal{S}_{n+1}$-modules

$$\begin{array}{ccc} P_{n+1} & \xrightarrow{j_{n+1}} & H^1(C_{n+1}) \\ \downarrow{\scriptstyle \iota_n} & & \downarrow \\ P_n \oplus Q_n & \xrightarrow{j_n} & H^1(\tilde{C}_n) \end{array}$$

in which the right hand vertical map is induced by (5.31). Note here that Nakayama's Lemma implies any map j_{n+1} in such a diagram is surjective since $j_n \circ \iota_n$ is surjective and the kernel of the projection $\mathcal{S}_{n+1} \to \mathcal{S}_n$ lies in the Jacobson radical of $\mathcal{S}_{n+1}$. The argument of Lemma 2.35 (i) now implies that C_{n+1} has a resolution $P_{n+1} \xrightarrow{d_{n+1}} P_{n+1}$, in which the first term is placed in degree zero and the isomorphism $\mathrm{cok}(\phi_{n+1}) \cong H^1(C_{n+1})$ is induced by j_{n+1}, and the descent isomorphism (5.31) is induced by a commutative diagram of $\mathcal{S}_{n+1}$-modules

$$\begin{array}{ccc} P_{n+1} & \xrightarrow{d_{n+1}} & P_{n+1} \\ \downarrow{\scriptstyle i'_n} & & \downarrow{\scriptstyle i_n} \\ P_n \oplus Q_n & \xrightarrow{d'_n} & P_n \oplus Q_n. \end{array}$$

In particular, since this morphism of complexes induces a quasi-isomorphism between the lower complex and the image under $- \otimes_{\mathcal{S}_{n+1}} \mathcal{S}_n$ of the upper complex, and the map $i_n \otimes_{\mathcal{S}_{n+1}} \mathcal{S}_n$ is bijective, the map i'_n must induce an isomorphism $(P_{n+1})_n \cong P_n \oplus Q_n$. The map i'_n therefore induces a surjective map

$$\bigwedge\nolimits_{\mathcal{S}_{n+1}}^{t-1} P_{n+1}^* \twoheadrightarrow \big(\bigwedge\nolimits_{\mathcal{S}_{n+1}}^{t-1} P_{n+1}^*\big) \otimes_{\mathcal{S}_{n+1}} \mathcal{S}_n \cong \bigwedge\nolimits_{\mathcal{S}_n}^{t-1} (P_n \oplus Q_n)^*$$

and we fix a pre-image f_{n+1} of $\widetilde{f}_n$ under this map. Having inductively constructed modules P_n and elements f_n, we now claim that the properties (i) and (ii) are satisfied by the elements

$$\Psi_{F_n} := \rho_n(f_n) \in \bigwedge\nolimits_{\mathcal{S}_n}^{t-1} H^1_{\mathscr{F}}(F_n, \mathcal{T})^*.$$

Here ρ_n denotes the restriction map $\bigwedge_{\mathcal{S}_n}^{t-1} P_n^* \to \bigwedge_{\mathcal{S}_n}^{t-1} \ker(d_n)^* \cong \bigwedge_{\mathcal{S}_n}^{t-1} H^1_{\mathscr{F}}(F_n, \mathcal{T})^*$, in which the last isomorphism is induced by Proposition 3.45 (v).

It is enough to verify (i) with L/K taken to be F_{n+1}/F_n. The key observation for this is that, for each $y \in \bigwedge_{\mathcal{S}_{n+1}}^{t-1} P_{n+1}$, the composite map

$$\bigwedge\nolimits_{\mathcal{S}_n}^{t-1} (P_{n+1})_n \cong \bigwedge\nolimits_{\mathcal{S}_n}^{t-1} (P_{n+1})^{\mathrm{Gal}(F_{n+1}/F_n)} \cong \big(\bigwedge\nolimits_{\mathcal{S}_{n+1}}^{t-1} P_{n+1}\big)^{\mathrm{Gal}(F_{n+1}/F_n)} \hookrightarrow \bigwedge\nolimits_{\mathcal{S}_{n+1}}^{t-1} P_{n+1}$$

sends $\mathrm{N}^{t-1}_{\mathrm{Gal}(F_{n+1}/F_n)} y$ to $\mathrm{N}_{\mathrm{Gal}(F_{n+1}/F_n)} y$. It follows that

$$\mathrm{N}_{\mathrm{Gal}(F_{n+1}/F_n)} \cdot f_n(\mathrm{N}^t_{\mathrm{Gal}(F_{n+1}/F_n)} y) = f_{n+1}(\mathrm{N}_{\mathrm{Gal}(F_{n+1}/F_n)} y),$$

and this implies the equality in (i) since the assignment $\mathrm{N}^t_{\mathrm{Gal}(F_{n+1}/F_n)}y \mapsto \mathrm{N}_{\mathrm{Gal}(F_{n+1}/F_n)}y$ restricts to give the map $\nu^t_{F_{n+1}/F_n}$ on $\bigcap^t_{\mathcal{S}_n} H^1_{\mathscr{F}}(F_n, \mathcal{T})$.

To verify (ii) we fix a divisor $\mathfrak{m}$ of $\mathfrak{n}$. We regard $H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{m}), \mathcal{A})$ as a submodule of $H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}), \mathcal{A})$ (just as in (4.4)) and write $\widetilde{\Phi}_{\mathfrak{m}}$ for the image of $\widetilde{\Phi}$ under the induced restriction map

$$\bigwedge\nolimits^{t-1}_{\Lambda[G_{\mathfrak{n}}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{n}), \mathcal{A})^* \to \bigwedge\nolimits^{t-1}_{\Lambda[G_{\mathfrak{m}}]} H^1_{\tilde{\mathscr{F}}}(k(\mathfrak{m}), \mathcal{A})^*.$$

Then, since $\Psi_{k(\mathfrak{m})}$ is the restriction of $\Psi_{k(\mathfrak{n})} = \rho_n(f_1)$ and $\widetilde{\Phi}$ is a pre-image of Φ under the restriction map (5.29), our explicit choice of f_1 implies that

$$D'_{\mathfrak{m}}\pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(\Psi_{k(\mathfrak{m})}(c_{k(\mathfrak{m})})) = D'_{\mathfrak{m}}\widetilde{\Phi}_{\mathfrak{m}}(\pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(c_{k(\mathfrak{m})})) = \widetilde{\Phi}_{\mathfrak{m}}(D'_{\mathfrak{m}}\pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(c_{k(\mathfrak{m})})).$$

Hence one has

$$\begin{aligned}
&(\nu^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}})^{-1}(D'_{\mathfrak{m}}\pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(\Psi_{k(\mathfrak{m})}(c_{k(\mathfrak{m})})))\\
&= (\nu^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}})^{-1}(\widetilde{\Phi}_{\mathfrak{m}}(D'_{\mathfrak{m}}\pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(c_{k(\mathfrak{m})})))\\
&= (\nu^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}})^{-1}((\widetilde{\Phi}_{\mathfrak{m}} \circ \nu^t_{k(\mathfrak{m}),\tilde{\mathscr{F}}} \circ (\nu^t_{k(\mathfrak{m}),\tilde{\mathscr{F}}})^{-1})(D'_{\mathfrak{m}}\pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(c_{k(\mathfrak{m})})))\\
&= \Phi((\nu^t_{k(\mathfrak{m})/k,\tilde{\mathscr{F}}})^{-1}(D'_{\mathfrak{m}}\pi^1_{k(\mathfrak{m}),\tilde{\mathscr{F}}}(c_{k(\mathfrak{m})}))),
\end{aligned}$$

where the last equality uses the fact $\widetilde{\Phi}_{\mathfrak{m}} \circ \nu^t_{k(\mathfrak{m})/k,\tilde{\mathscr{F}}} = \nu^1_{k(\mathfrak{m})/k,\tilde{\mathscr{F}}} \circ \Phi$ on $\bigcap^t_{\Lambda} H^1_{\tilde{\mathscr{F}}^{\mathfrak{m}}}(k, \mathcal{A})$. This verifies the required property (ii). □

A collection $\Psi = (\Psi_K)_{K\in\Omega}$ of maps that satisfy the relations in Proposition 5.28 (i) is often referred to as a 'Perrin-Riou functional' (see, for example, [68, Def. 8.1.2]). The key observation, made independently by Rubin in [91, § 6] and by Perrin-Riou in [87, § 1.2.3], is that any such collection gives rise to a well-defined map of $\mathcal{R}[\![\mathcal{G}_{\mathcal{K}}]\!]$-modules

$$\mathrm{ES}^t(\mathfrak{F}_{\mathrm{rel}}(\mathcal{T})) \to \mathrm{ES}^1(\mathfrak{F}_{\mathrm{rel}}(\mathcal{T})), \quad (c_K)_{K\in\Omega} \mapsto (\Psi_K(c_K))_{K\in\Omega}$$

where $\mathfrak{F}_{\mathrm{rel}}(\mathcal{T})$ is as defined in Example 4.9. By precisely the same argument, one verifies that this result remains valid after replacing $\mathfrak{F}_{\mathrm{rel}}(\mathcal{T})$ by any family $\mathfrak{F}$ satisfying Hypothesis 4.1 and this observation then combines with Proposition 5.28 to prove the following reduction result for Theorem 5.27.

(5.32) Lemma. *To prove Theorem 5.27 it is enough to consider the case $t = 1$.*

Proof. We first show that, for each $\mathfrak{n} \in \mathcal{N}$, the element $\kappa(c)_{\mathfrak{n}}$ belongs to $\bigcap^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(K, \mathcal{A})$. For this, Corollary 2.24 implies it suffices to show that, for every $\Phi \in \bigwedge^{t-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}_{\mathfrak{n}}}(k, \mathcal{A})^*$ one has

$$\Phi(\kappa(c)_{\mathfrak{n}}) = \sum\nolimits_{\tau\in\mathfrak{S}(\mathfrak{n})} \mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{q}\in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{q})}_{\tau(\mathfrak{q})}\big) \cdot \Phi(\kappa'(c)_{\mathfrak{d}_\tau}) \quad \in H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A}). \qquad (5.33)$$

However, by Proposition 5.28, there exists $\Psi \in \varprojlim_F \bigwedge^{t-1}_{\mathcal{R}[\mathcal{G}_F]} H^1_{\mathscr{F}}(F, \mathcal{T})^*$ such that

$$\Phi(\kappa'(c_{\mathfrak{d}})) = \kappa'(\Psi(c)_{\mathfrak{d}})$$

for every divisor $\mathfrak{d}$ of $\mathfrak{n}$. In particular, since $\Psi(c)$ belongs to $\mathrm{ES}^1(\mathfrak{F})$, the containment (5.33) is valid for all t if it is valid for $t = 1$.

We show next that, for every $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{q} \in V(\mathfrak{n})$, one has

$$v_{\mathfrak{q}}(\kappa(c)_{\mathfrak{n}}) = \psi^{\mathrm{fs}}_{\mathfrak{q}}(\kappa(c)_{\mathfrak{n}/\mathfrak{q}}).$$

For this we note that, since $\bigcap^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})$ is defined to be $(\bigwedge^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})^*)^*$, the element $\kappa(c)_{\mathfrak{n}}$ of $\bigcap^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})$ defines a map $\bigwedge^t_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})^* \to \mathbb{A}$. Given this, $v_{\mathfrak{q}}(\kappa(c)_{\mathfrak{n}})$ is the map

$$\bigwedge\nolimits^{t-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})^* \to \mathbb{A}, \quad \Phi \mapsto \kappa(c)_{\mathfrak{n}}(v_{\mathfrak{q}} \wedge \Phi).$$

In particular, if we identify $\bigcap^1_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})$ with $H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})$, and thereby regard $\Phi(\kappa(c)_\mathfrak{n})$ as an element of $H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})$, then we have

$$\kappa(c)_\mathfrak{n}(v_\mathfrak{q}\wedge\Phi) = (-1)^{t-1}\cdot v_\mathfrak{q}(\Phi(\kappa(c)_\mathfrak{n})).$$

Similarly, one finds that $\psi^{\mathrm{fs}}_\mathfrak{q}(\kappa(c)_{\mathfrak{n}/\mathfrak{q}}) \in \bigcap^{t-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})$ identifies with the map

$$\bigwedge\nolimits^{t-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})^* \to \Lambda, \quad \Phi\mapsto \kappa(c)_{\mathfrak{n}/\mathfrak{q}}(\psi^{\mathrm{fs}}_\mathfrak{q}\wedge\Phi) = (-1)^{t-1}\cdot\psi^{\mathrm{fs}}_\mathfrak{q}(\Phi(\kappa(c)_{\mathfrak{n}/\mathfrak{q}})).$$

To verify the claimed equality $v_\mathfrak{q}(\kappa(c)_\mathfrak{n}) = \psi^{\mathrm{fs}}_\mathfrak{q}(\kappa(c)_{\mathfrak{n}/\mathfrak{q}})$, it is thus enough to show that, for every $\Phi\in\bigwedge^{t-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})^*$, one has

$$v_\mathfrak{q}(\Phi(\kappa(c)_\mathfrak{n})) = \psi^{\mathrm{fs}}_\mathfrak{q}(\Phi(\kappa(c)_{\mathfrak{n}/\mathfrak{q}})) \tag{5.34}$$

Moreover, by using Proposition 5.28, we can choose an element Ψ of $\varprojlim_F \bigwedge^{a-1}_{\mathcal{R}[\mathcal{G}_F]} H^1_{\mathscr{F}}(F,\mathcal{T})^*$ with both $\Phi(\kappa(c)_\mathfrak{n}) = \kappa(\Psi(c))_\mathfrak{n}$ and $\Phi(\kappa(c)_{\mathfrak{n}/\mathfrak{q}}) = \kappa(\Psi(c))_{\mathfrak{n}/\mathfrak{q}}$. In particular, since $\Psi(c)$ belongs to $\mathrm{ES}^1(\mathfrak{F})$, the required equality (5.34) will therefore follow if Theorem 5.27 is known to be valid in the case $t=1$. □

5.3.3. The proof of Theorem 5.27 in rank one

In this section we shall prove (in Propositions 5.40 and 5.44) the result of Theorem 5.27 in the special case $t=1$. In view of Lemma 5.32, the results presented here will therefore also complete the proof of Theorem 5.27 in the general case.

In the first result, we use an approach of Kato (from [59, § 2]), to show that Euler systems for Nekovář structures satisfy the 'congruence condition' discussed in [92, § 4.8]. For each $\mathfrak{q}\in\mathcal{Q}$ and $K\in\Omega$, we write

$$\mathrm{loc}_{K,\mathfrak{q}}\colon H^1_{\mathfrak{F}}(K,\mathcal{T})\to \bigoplus\nolimits_{v\mid\mathfrak{q}} H^1(K_v,\mathcal{T})\cong H^1(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_K])$$

for the natural localisation map. We will also often use the fact that, for $\mathfrak{n}\in\mathcal{N}$ and $\mathfrak{q}\in V(\mathfrak{n})$, the extension $k(\mathfrak{n})/k(\mathfrak{n}/\mathfrak{q})$ is totally ramified at all places above $\mathfrak{q}$, and hence that for $q\in\{0,1\}$ there is a natural isomorphism $H^q_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])\cong H^q_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}])$.

(5.35) Proposition. *Assume $\mathfrak{F}$ satisfies Hypothesis 4.1 and that $\mathcal{K}$ and $\mathcal{T}$ satisfy Hypothesis 4.16. Then, for every $\mathfrak{n}\in\mathcal{N}$, $\mathfrak{q}\in V(\mathfrak{n})$, and $c\in\mathrm{ES}^1(\mathfrak{F})$, the following claims are valid.*

(i) $\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1}) - \mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1})\in[k(\mathfrak{q}):k(1)]\cdot\mathcal{R}[G_{\mathfrak{n}/\mathfrak{q}}]$,

(ii) $\mathrm{loc}_{k(\mathfrak{n}),\mathfrak{q}}(c_{k(\mathfrak{n})})$ *and* $\mathrm{loc}_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}}(c_{k(\mathfrak{n}/\mathfrak{q})})$ *belong to* $H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])\cong H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}])$.

(iii) One has an equality

$$\mathrm{loc}_{k(\mathfrak{n}),\mathfrak{q}}(c_{k(\mathfrak{n})}) = \left(\frac{\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1}) - \mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1})}{[k(\mathfrak{q}):k(1)]}\right)\cdot\mathrm{loc}_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}}(c_{k(\mathfrak{n}/\mathfrak{q})}).$$

Proof. Claim (i) is true since $\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1}) - \mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1})$ belongs to $(\mathrm{N}\mathfrak{q}-1)\cdot\mathcal{R}$ and $\mathrm{N}\mathfrak{q}-1$ is divisible by $[k(\mathfrak{q}):k(1)]$.

We next note $\mathrm{im}(\mathrm{loc}_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}})\subseteq H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}])$ as $\mathfrak{q}\notin S(\mathscr{F}_{k(\mathfrak{n}/\mathfrak{q})})$. In particular, since $c_{k(\mathfrak{n}/\mathfrak{q})}\in H^1_{\mathfrak{F}}(k(\mathfrak{n}/\mathfrak{q}),\mathcal{T})$, one has $\mathrm{loc}_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}}(c_{k(\mathfrak{n}/\mathfrak{q})})\in H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}])$. To prove (ii) it is therefore sufficient to prove that $\mathrm{loc}_{k(\mathfrak{n}),\mathfrak{q}}(c_{k(\mathfrak{n})})$ belongs to $H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$.

To do this, will make use of the fact that, by the assumed validity of Hypothesis 4.16 (ii), the field $\mathcal{K}$ contains a $\mathbb{Z}_p$-extension k_∞ of k in which no finite place splits completely. Set $k(\mathfrak{n})_\infty\coloneqq k_\infty\cdot k(\mathfrak{n})$ and, writing k_n for the n-th layer of k_n/k, also $k(\mathfrak{n})_n\coloneqq k_n\cdot k(\mathfrak{n})$. We define $\mathcal{S}\coloneqq\mathcal{R}[\![\mathcal{G}_{k(\mathfrak{n})_\infty}]\!]$ to be the relevant Iwasawa algebra and consider $\mathcal{T}\otimes_\mathcal{R}\mathcal{S}$, which is a free $\mathcal{S}$-module with G_k-action given by $\sigma\cdot(a\otimes b)\coloneqq(\sigma a)\otimes(b\overline{\sigma}^{-1})$ for every $\sigma\in G_k$. (Here $\overline{\sigma}$ denotes the image of σ in $\mathcal{G}_{k_\infty}$.) Then Shapiro's lemma implies

$$H^1(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S})\cong\varprojlim\nolimits_{n\in\mathbb{N}} H^1(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})_n}]),$$

where the limit is taken with respect to corestriction maps, and so the Euler system norm relations imply that the family $(\mathrm{loc}_{k(\mathfrak{n})_n,\mathfrak{q}}(c_{k(\mathfrak{n})_n}))_{n\in\mathbb{N}}$ defines an element of $H^1(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S})$. Hence, if we can prove that the natural composite map

$$\varprojlim_{n\in\mathbb{N}} H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})_n}]) \cong H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S}) \to H^1(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S})$$

is bijective, then one has $\mathrm{loc}_{k(\mathfrak{n}),\mathfrak{q}}(c_{k(\mathfrak{n})}) \in H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$, as required. If $\mathcal{R}$ is finitely generated over $\mathbb{Z}_p$, then the bijectivity of the displayed map is proved by Rubin in [92, Prop. B.3.4], and we now adapt the argument of loc. cit. to our more general setting.

For this, we write F for the completion of $k(\mathfrak{n})$ at a place above $\mathfrak{q}$ and F_n for the n-th layer of the (unique) unramified $\mathbb{Z}_p$-extension of F. The inertia subgroup $\mathcal{I} \coloneqq \mathcal{I}_{F_n} \subseteq G_{F_n}$ is then independent of n because F_n/F is unramified. Since $\mathcal{I}$ acts trivially on $\mathcal{T}$, the relevant case of the inflation-restriction sequence reads

$$0 \to H^1_f(F_n,\mathcal{T}) \to H^1(F_n,\mathcal{T}) \to H^1(\mathcal{I},\mathcal{T})^{G_{F_n}/\mathcal{I}} = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{I}^{(p)},\mathcal{T})^{G_{F_n}}, \tag{5.36}$$

where $\mathcal{I}^{(p)}$ denotes the pro-p completion of $\mathcal{I}$. Local class field theory implies that $\mathcal{I}^{(p)}$ is a finitely generated $\mathbb{Z}_p$-module and so $X_n \coloneqq \mathrm{Hom}_{\mathrm{cont}}(\mathcal{I}^{(p)},\mathcal{T})^{G_{F_n}}$ is a finitely generated $\mathcal{R}$-module. It follows that each X_n is Noetherian (because $\mathcal{R}$ is) and p-torsion free (because $\mathcal{T}$ is by Hypothesis 4.16 (ii)). In particular, the ascending chain $(X_n)_{n\in\mathbb{N}}$ needs to become stationary and so there is $m \in \mathbb{N}$ such that $X_n = X_m$ for all $n \geq m$. This implies that, for each $n \geq m$, the norm map $X_{n+1} \to X_n$ is induced by multiplication by p, and hence that $\varprojlim_{n\in\mathbb{N}} X_n$ vanishes. Taking the limit (over n) of the exact sequence (5.36) therefore leads to an isomorphism $\varprojlim_{n\in\mathbb{N}} H^1_f(F_n,\mathcal{T}) \cong \varprojlim_{n\in\mathbb{N}} H^1(F_n,\mathcal{T})$. Since (by assumption) the set $\Sigma_\mathfrak{q}$ of places $k(\mathfrak{n})_\infty$ above $\mathfrak{q}$ is finite, this in turn induces an isomorphism

$$H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S}) \cong \varprojlim_{n\in\mathbb{N}} \bigoplus_{v\in\Sigma_\mathfrak{q}} H^1_f(k(\mathfrak{n})_{n,v},\mathcal{T}) \to \varprojlim_{n\in\mathbb{N}} \bigoplus_{v\in\Sigma_\mathfrak{q}} H^1(k(\mathfrak{n})_{n,v},\mathcal{T}) \cong H^1(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S}),$$

as required to complete the proof of (ii).

Before proving (iii), we claim that the group $H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S})$ is p-torsion free. To see this, we write $\mathcal{S}_v \coloneqq \mathcal{R}[\![\mathrm{Gal}(k(\mathfrak{n})_{\infty,v}/k_\mathfrak{q})]\!]$ for every $v \in \Sigma_\mathfrak{q}$ and note that Shapiro's Lemma gives a direct sum decomposition

$$H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S}) \cong \bigoplus_{v\in\Sigma_\mathfrak{q}} H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S}_v) \cong \bigoplus_{v\in\Sigma_\mathfrak{q}} H^1(\mathbb{F}_v,\mathcal{T}),$$

where $\mathbb{F}_v$ denotes the residue field of v. However, every finite extension of $\mathbb{F}_v$ has degree coprime to p so that Frob_v must act trivially on $\mathcal{T}$. It follows that $H^1(\mathbb{F}_v,\mathcal{T}) \cong \mathcal{T}/(1-\mathrm{Frob}_v) \cong \mathcal{T}$ and hence that $H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S})$ is p-torsion free because $\mathcal{T}$ is by Hypothesis 4.16 (ii).

Turning now to (iii), we observe (ii) implies (iii) is true if, in $H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S})$, one has

$$d_\mathfrak{n} = \left(\frac{\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1}) - \mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1})}{[k(\mathfrak{q}):k(1)]}\right)\cdot d_{\mathfrak{n}/\mathfrak{q}}.$$

In addition, since $H^1_f(k_\mathfrak{q},\mathcal{T}\otimes_\mathcal{R}\mathcal{S})$ is p-torsion free, this equality will follow if we can prove that

$$[k(\mathfrak{q}):k(1)]\cdot d_\mathfrak{n} = \big(\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1}) - \mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1})\big)\cdot d_{\mathfrak{n}/\mathfrak{q}}. \tag{5.37}$$

To do this, we we first note that the trace element $\mathrm{N}_{G_\mathfrak{q}}$ acts as multiplication by $[k(\mathfrak{q}):k(1)]$ on $H^1_f(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})_n}]) = H^1(\mathbb{F}_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})_n}])$, and hence that

$$[k(\mathfrak{q}):k(1)]\cdot\mathrm{loc}_{k(\mathfrak{n})_n,\mathfrak{q}}(c_{k(\mathfrak{n})_n}) = \mathrm{loc}_{k(\mathfrak{n})_n,\mathfrak{q}}(\mathrm{N}_{G_\mathfrak{q}}c_{k(\mathfrak{n})_n}) = \mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1})\cdot\mathrm{loc}_{k(\mathfrak{n}/\mathfrak{q})_n,\mathfrak{q}}(c_{k(\mathfrak{n}/\mathfrak{q})_n}).$$

On the other hand, we can compute from

$$0 \to \mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})_n}] \xrightarrow{1-\mathrm{Frob}_v} \mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})_n}] \to H^1(\mathbb{F}_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})_n}]) \to 0$$

that $\det_{\mathcal{R}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})_n}]}(1-\mathrm{Frob}_v \mid \mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})_n}]) = \mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1})$ annihilates $H^1(\mathbb{F}_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})_n}])$. This proves (5.37), and hence also (iii). □

In order to prove the 'finite-singular relations', we recall certain maps introduced by Kato in [59, § 4]. To do this, we fix $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{q} \in \mathcal{Q}$, and write $\Psi^{\mathrm{fs}}_{\mathfrak{n},\mathfrak{q}}$ for the composite

$$H^1_f(k_\mathfrak{q}, \mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}]) \to H^1_f(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}]) \xrightarrow{(5.5)} (\mathcal{A}[G_{k(\mathfrak{n})}])^{\mathcal{I}_{k(\mathfrak{n}),\mathfrak{q}}}/(\mathrm{Frob}_\mathfrak{q} - 1) \xrightarrow{c_{1-\mathrm{Frob}_\mathfrak{q}}} H^0_f(k_\mathfrak{q}, \mathcal{A}[G_{k(\mathfrak{n})}]).$$

Here the first map is induced by the projection $\mathcal{T} \to \mathcal{A}$, we have written $\mathcal{I}_{k(\mathfrak{n}),\mathfrak{q}} \subseteq \mathcal{G}_K$ for the inertia subgroup at $\mathfrak{q}$ (so $\mathcal{I}_{k(\mathfrak{n}),\mathfrak{q}}$ identifies with $G_\mathfrak{q}$ if $\mathfrak{q} \in V(\mathfrak{n})$ and is trivial otherwise), and $c_{1-\mathrm{Frob}_\mathfrak{q}}$ is the 'cofactor map' of multiplication by $1 - \mathrm{Frob}_\mathfrak{q}$ on $\mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}/\mathcal{I}_{k(\mathfrak{n}),\mathfrak{q}}]$ (as defined in the proof of Lemma 5.3).
If $\mathfrak{q} \in V(\mathfrak{n})$, then we also have a well-defined 'evaluation' map

$$\mathrm{ev}_{\sigma_\mathfrak{q}} \colon H^1(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}]) \to \big(\mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}]/(\sigma_\mathfrak{q} - 1)\big)^{G_{\kappa_\mathfrak{q}}} \cong H^0_f(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}]), \quad [\xi] \mapsto \xi(\sigma_\mathfrak{q}).$$

that we can use to define a map

$$V_{\mathfrak{n},\mathfrak{q}} \colon H^1(k_\mathfrak{q}, \mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}]) \to H^1(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}]) \xrightarrow{\mathrm{ev}_{\sigma_\mathfrak{q}}} H^0_f(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}]) \cong H^0_f(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}],$$

where the first arrow is again induced by $\mathcal{T} \to \mathcal{A}$ and the last isomorphism holds because $k(\mathfrak{n})/k(\mathfrak{n}/\mathfrak{q})$ is totally ramified at $\mathfrak{q}$. Finally, we write $\iota_{\mathfrak{n},\mathfrak{q}} \colon H^0(k_\mathfrak{q}, \mathcal{A}) \hookrightarrow H^0(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}])$ for the map induced by the assignment $\mathcal{A} \to \mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}]$ sending each a to $a \cdot \mathrm{N}_{\mathcal{G}_{k(\mathfrak{n})}}$.
The maps $\Psi^{\mathrm{fs}}_{\mathfrak{n},\mathfrak{q}}$ and $V_{\mathfrak{n},\mathfrak{q}}$ are related to the maps $\psi^{\mathrm{fs}}_\mathfrak{q}$ and $v_\mathfrak{q}$ in the following way.

(5.38) Lemma. *Write $h \colon \mathcal{A}/(\tau - 1) \cong \mathbb{A}$ for the isomorphism fixed in Hypothesis 4.14 (ii). Then, for every $c \in \mathrm{ES}^1(\mathfrak{F})$, $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{q} \in \mathcal{Q}$ one has*

$$(\Psi^{\mathrm{fs}}_{\mathfrak{n},\mathfrak{q}} \circ \mathrm{loc}_{k(\mathfrak{n}),\mathfrak{q}})(D'_\mathfrak{n} c_{k(\mathfrak{n})}) = (\iota_{\mathfrak{n},\mathfrak{q}} \circ h^{-1} \circ \check{\psi}^{\mathrm{fs}}_\mathfrak{q})(\kappa'(c)_\mathfrak{n})$$

and, if $\mathfrak{q} \in V(\mathfrak{n})$, also

$$(V_{\mathfrak{n},\mathfrak{q}} \circ \mathrm{loc}_{k(\mathfrak{n}),\mathfrak{q}})(D'_\mathfrak{n} c_{k(\mathfrak{n})}) = (\iota_{\mathfrak{n},\mathfrak{q}} \circ c^{-1}_{1-\tau} \circ h^{-1} \circ \check{v}_\mathfrak{q})(\kappa'(c)_\mathfrak{n}).$$

Proof. This follows directly from comparing the definitions of the involved maps. □

Write $\delta_{K,\mathfrak{q}} \colon H^0(k_\mathfrak{q}, \mathcal{A}[\mathcal{G}_K]) \to H^1(k_\mathfrak{q}, \mathfrak{a}_i\mathcal{T}[\mathcal{G}_K])$ for the connecting homomorphism arising from the tautological short exact sequence $0 \to \mathfrak{a}_i\mathcal{T}[\mathcal{G}_K] \to \mathcal{T}[\mathcal{G}_K] \to \mathcal{A}[\mathcal{G}_K] \to 0$. Then the following result allows for explicit computation of the maps $\delta_{k(\mathfrak{n}),\mathfrak{q}} \circ \Psi^{\mathrm{fs}}_{\mathfrak{n},\mathfrak{q}}$ and $\delta_{k(\mathfrak{n}),\mathfrak{q}} \circ V_{\mathfrak{n},\mathfrak{q}}$.

(5.39) Lemma. *Let $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{q} \in \mathcal{Q}$. Then the following claims are valid.*

(i) For every $y \in H^1_f(k_\mathfrak{q}, \mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])^{\mathcal{I}_{K,\mathfrak{q}}}$ we have

$$(\delta_{k(\mathfrak{n}),\mathfrak{q}} \circ \Psi^{\mathrm{fs}}_{\mathfrak{n},\mathfrak{q}})(y) = -\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}) \cdot y$$

(Here we are using the fact that the definition of $\mathcal{Q}$ ensures $\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}) \in \mathfrak{a}_i[\mathcal{G}_{k(\mathfrak{n})}]$ and so $\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}) \cdot y$ can be regarded as an element of $H^1_f(k_\mathfrak{q}, \mathfrak{a}_i\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$.)

(ii) If $\mathfrak{q} \in V(\mathfrak{n})$, then for every $y \in H^1_f(k_\mathfrak{q}, \mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$ we have

$$(\delta_{k(\mathfrak{n}),\mathfrak{q}} \circ V_{\mathfrak{n},\mathfrak{q}})(y) = (\sigma_\mathfrak{q} - 1) \cdot y.$$

(Here we are using that $\sigma_\mathfrak{q}$ acts trivially on $\mathcal{A}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}]$ so that $(\sigma_\mathfrak{q} - 1)a$ can be regarded an element of $H^1_f(k_\mathfrak{q}, \mathfrak{a}_i\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$.)

(iii) Assume $\mathfrak{F}$ satisfies Hypothesis 4.1 and that $\mathcal{K}$ and $\mathcal{T}$ satisfy Hypothesis 4.16. Then, for every $c \in \mathrm{ES}^1(\mathfrak{F})$, $\mathfrak{n} \in \mathcal{N}$, and $\mathfrak{q} \in V(\mathfrak{n})$ one has

$$(\delta_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}} \circ \Psi^{\mathrm{fs}}_{\mathfrak{n}/\mathfrak{q},\mathfrak{q}})(D'_{\mathfrak{n}/\mathfrak{q}} c_{k(\mathfrak{n}/\mathfrak{q})}) = (\delta_{k(\mathfrak{n}),\mathfrak{q}} \circ V_{\mathfrak{n},\mathfrak{q}})(D'_\mathfrak{n} c_{k(\mathfrak{n})})$$

in $H^1_f(k_\mathfrak{q}, \mathfrak{a}_i\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}]) \cong H^1_f(k_\mathfrak{q}, \mathfrak{a}_i\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$.

Proof. To prove (i), we take $y \in H^1_f(k_\mathfrak{q}, \mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])^{\mathcal{I}_{K,\mathfrak{q}}}$ and note $\Psi^{\mathrm{fs}}_{\mathfrak{n},\mathfrak{q}}(y) = c_{\mathrm{Frob}_\mathfrak{q}-1} \cdot y(\mathrm{Frob}_\mathfrak{q})$. The cofactor map $c_{\mathrm{Frob}_\mathfrak{q}-1}$ satisfies

$$\begin{aligned}(\mathrm{Frob}_\mathfrak{q} - 1)c_{\mathrm{Frob}_\mathfrak{q}-1} &= -(1 - \mathrm{Frob}_\mathfrak{q})c_{\mathrm{Frob}_\mathfrak{q}-1} \\ &= -\det\nolimits_{\mathbb{A}[\mathcal{G}_{k(\mathfrak{n})}/\mathcal{I}_{k(\mathfrak{n}),\mathfrak{q}}]}(\mathrm{Frob}_\mathfrak{q} - 1 \mid \mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}/\mathcal{I}_{k(\mathfrak{n}),\mathfrak{q}}]) \\ &= -\mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q} \cdot \mathrm{Frob}_\mathfrak{q}) \quad \in \mathbb{A}[\mathcal{G}_{k(\mathfrak{n})}/\mathcal{I}_{k(\mathfrak{n}),\mathfrak{q}}].\end{aligned}$$

This combines with the definition of $\delta_{k(\mathfrak{n}),\mathfrak{q}}$ to imply that

$$\begin{aligned}(\delta_{k(\mathfrak{n}),\mathfrak{q}}\circ\Psi^{\rm fs}_{\mathfrak{n},\mathfrak{q}})(y)(\mathrm{Frob}_\mathfrak{q}) &= (\mathrm{Frob}_\mathfrak{q}-1)\cdot c_{\mathrm{Frob}_\mathfrak{q}-1}\cdot y(\mathrm{Frob}_\mathfrak{q})\\ &= -\mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q})\cdot y(\mathrm{Frob}_\mathfrak{q})\\ &= (-\mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q})\cdot y)(\mathrm{Frob}_\mathfrak{q}).\end{aligned}$$

As an element of $H^1_f(k_\mathfrak{q},\mathcal{A}[G_{k(\mathfrak{n})}])$, the class $(\delta_{k(\mathfrak{n}),\mathfrak{q}}\circ\Psi^{\rm fs}_{\mathfrak{n},\mathfrak{q}})(y)$ is uniquely determined by its value on $\mathrm{Frob}_\mathfrak{q}$. The above calculation therefore verifies (i).
To prove (ii), we take $y\in H^1(k_\mathfrak{q},\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$ and similarly compute that

$$(\delta_{k(\mathfrak{n}),\mathfrak{q}}\circ V_{\mathfrak{n},\mathfrak{q}})(y)(\mathrm{Frob}_\mathfrak{q}) = (\mathrm{Frob}_\mathfrak{q}-1)\cdot y(\widetilde{\sigma_\mathfrak{q}}),$$

where $\widetilde{\sigma_\mathfrak{q}}\in G_k$ denotes a lift of $\sigma_\mathfrak{q}$. By the definition of the action of $G_\mathfrak{q}$ on $H^1_f(k_\mathfrak{q},\mathfrak{a}_i\mathcal{T}[G_\mathfrak{n}])$ and by repeatedly using the cocycle property we may calculate that

$$\begin{aligned}((\sigma_\mathfrak{q}-1)y)(\mathrm{Frob}_\mathfrak{q}) &= \widetilde{\sigma_\mathfrak{q}}y(\widetilde{\sigma_\mathfrak{q}}^{-1}\mathrm{Frob}_\mathfrak{q}\widetilde{\sigma_\mathfrak{q}})-y(\mathrm{Frob}_\mathfrak{q}) = \big(y(\mathrm{Frob}_\mathfrak{q}\widetilde{\sigma_\mathfrak{q}})+\widetilde{\sigma_\mathfrak{q}}y(\widetilde{\sigma_\mathfrak{q}}^{-1})\big)-y(\mathrm{Frob}_\mathfrak{q})\\ &= \big(\mathrm{Frob}_\mathfrak{q}y(\widetilde{\sigma_\mathfrak{q}})+y(\mathrm{Frob}_\mathfrak{q})\big)-y(\widetilde{\sigma_\mathfrak{q}})-y(\mathrm{Frob}_\mathfrak{q}) = (\mathrm{Frob}_\mathfrak{q}-1)y(\widetilde{\sigma_\mathfrak{q}}).\end{aligned}$$

We have therefore proved $(\delta_{k(\mathfrak{n}),\mathfrak{q}}\circ V_{\mathfrak{n},\mathfrak{q}})(y)(\mathrm{Frob}_\mathfrak{q}) = ((\sigma_\mathfrak{q}-1)y)(\mathrm{Frob}_\mathfrak{q})$, and this implies (ii).
To verify (iii), we note $|G_\mathfrak{q}|$ annihilates the kernel of the map $H^1_f(k_\mathfrak{q},\mathfrak{a}_i\mathcal{T}[G_\mathfrak{n}])\to H^1(k_\mathfrak{q},\mathcal{T}[G_\mathfrak{n}])$ (which identifies with a submodule of $\mathcal{A}[G_\mathfrak{n}]$), and so Proposition 5.35 (iii) implies that

$$|G_\mathfrak{q}|c_\mathfrak{n} = (\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1})-\mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1}))c_{\mathfrak{n}/\mathfrak{q}}\in H^1_f(k_\mathfrak{q},\mathfrak{a}_i\mathcal{T}[G_{\mathfrak{n}/\mathfrak{q}}]).$$

Combining this with the relation $(\sigma_\mathfrak{q}-1)D_\mathfrak{q} = |G_\mathfrak{q}|-\mathrm{N}_{G_\mathfrak{q}}$ and the result of (i), we then compute

$$\begin{aligned}&(\delta_{k(\mathfrak{n}),\mathfrak{q}}\circ V_{\mathfrak{n},\mathfrak{q}})(D'_\mathfrak{n}c_{k(\mathfrak{n})})\\ &= (\sigma_\mathfrak{q}-1)D'_\mathfrak{n}c_{k(\mathfrak{n})}\\ &= (|G_\mathfrak{q}|-\mathrm{N}_{G_\mathfrak{q}})D'_{\mathfrak{n}/\mathfrak{q}}c_{k(\mathfrak{n})}\\ &= |G_\mathfrak{q}|D'_{\mathfrak{n}/\mathfrak{q}}c_{k(\mathfrak{n})}-\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1})D'_{\mathfrak{n}/\mathfrak{q}}c_{k(\mathfrak{n}/\mathfrak{q})}\\ &= (\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1})-\mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1}))D'_{\mathfrak{n}/\mathfrak{q}}c_{\mathfrak{n}/\mathfrak{q}}-\mathrm{Eul}_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}^{-1})D'_{\mathfrak{n}/\mathfrak{q}}c_{k(\mathfrak{n}/\mathfrak{q})}\\ &= -\mathrm{Eul}_\mathfrak{q}(\mathrm{N}\mathfrak{q}\cdot\mathrm{Frob}_\mathfrak{q}^{-1})D'_{\mathfrak{n}/\mathfrak{q}}c_{\mathfrak{n}/\mathfrak{q}}\\ &= (\delta_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}}\circ\Psi^{\rm fs}_{\mathfrak{n},\mathfrak{q}})(D'_{\mathfrak{n}/\mathfrak{q}}c_{k(\mathfrak{n}/\mathfrak{q})}).\end{aligned}$$

where the final equality follows from (i). This proves the equality in (iii). □

We can now prove the 'finite-singular relations' in Theorem 5.27 in the case $a=1$.

(5.40) Proposition. *Assume $\mathfrak{F}$ satisfies Hypothesis 4.1 and that $\mathcal{K}$ and $\mathcal{T}$ satisfy Hypothesis 4.16. Then, for every $c\in\mathrm{ES}^1(\mathfrak{F})$, $\mathfrak{n}\in\mathcal{N}$, and $\mathfrak{q}\in V(\mathfrak{n})$ one has*

$$\check{v}_\mathfrak{q}(\kappa'(c)_\mathfrak{n}) = \check{\psi}^{\rm fs}_\mathfrak{q}(\kappa'(c)_{\mathfrak{n}/\mathfrak{q}})\in\mathbb{A}.$$

Proof. Since h and $c_{1-\tau}$ are bijective and $\iota_{\mathfrak{n},\mathfrak{q}}$ is injective, the claimed equality is valid if, in $H^0_f(k_\mathfrak{q},\mathcal{A}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}])\cong H^0_f(k_\mathfrak{q},\mathcal{A}[\mathcal{G}_{k(\mathfrak{n})}])$, one has

$$(\iota_{\mathfrak{n}/\mathfrak{q},\mathfrak{q}}\circ h^{-1}\circ\check{\psi}^{\rm fs}_\mathfrak{q})(\kappa'(c)_{\mathfrak{n}/\mathfrak{q}}) = (\iota_{\mathfrak{n},\mathfrak{q}}\circ c^{-1}_{1-\tau}\circ h^{-1}\circ\check{v}_\mathfrak{q})(\kappa'(c)_\mathfrak{n}).$$

Since Hypothesis 4.16 (ii) implies the connecting maps $\delta_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}}$ and $\delta_{k(\mathfrak{n}),\mathfrak{q}}$ are injective, it thus suffices to show that, in $H^1_f(k_\mathfrak{q},\mathfrak{a}_i\mathcal{T}[\mathcal{G}_{k(\mathfrak{n}/\mathfrak{q})}])\cong H^1_f(k_\mathfrak{q},\mathfrak{a}_i\mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$, one has

$$(\delta_{k(\mathfrak{n}/\mathfrak{q}),\mathfrak{q}}\circ\iota_{\mathfrak{n}/\mathfrak{q},\mathfrak{q}}\circ h^{-1}\circ\check{\psi}^{\rm fs}_\mathfrak{q})(\kappa'(c)_{\mathfrak{n}/\mathfrak{q}}) = (\delta_{k(\mathfrak{n}),\mathfrak{q}}\circ\iota_{\mathfrak{n},\mathfrak{q}}\circ c^{-1}_{1-\tau}\circ h^{-1}\circ\check{v}_\mathfrak{q})(\kappa'(c)_\mathfrak{n}).$$

It is then enough to note that the latter equality can be verified by combining Lemmas 5.38 and 5.39 (iii). □

It now only remains to show that the explicit linear combination of elements $\kappa'(c)_\mathfrak{d}c_{k(\mathfrak{d})}$ that occurs in Definition 5.26 belongs to $H^1_{\mathscr{F}(\mathfrak{n})}(k,\mathcal{A})$, and for this we adapt the argument of Mazur and Rubin in [74, App. A]. This means that the key technical result we have to prove is the following.

(5.41) Lemma ([74, Th. A.4]). *Assume $\mathfrak{F}$ satisfies Hypothesis 4.1 and that $\mathcal{K}$ and $\mathcal{T}$ satisfy Hypothesis 4.16. Then, for every $c \in \mathrm{ES}^1(\mathfrak{F})$, $\mathfrak{n} \in \mathcal{N}$, and $\mathfrak{q} \in V(\mathfrak{n})$ one has*

$$\check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}(\kappa'(c)_{\mathfrak{n}}) = -\sum\nolimits_{\tau \in \mathfrak{S}_{\mathfrak{q}}(\mathfrak{n})} \mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{l} \in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{l})}_{\tau(\mathfrak{l})}\big) \cdot \check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}(\kappa'(c)_{\mathfrak{d}_\tau}),$$

where $\mathfrak{S}_{\mathfrak{q}}(\mathfrak{n})$ denotes the subset of $\mathfrak{S}(\mathfrak{n})$ comprising cycles τ for which $\tau(\mathfrak{q}) \neq \mathfrak{q}$.

Proof. Let $\mathfrak{S}_1(\mathfrak{n})$ denote the collection of all cycles in $\mathrm{Per}(V(\mathfrak{n}))$. Then for each $\tau \in \mathfrak{S}_1(\mathfrak{n})$ one has $\mathfrak{q} \notin V(\mathfrak{d}_\tau)$ and so Proposition 5.40 shows that $\check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}(\kappa'(c)_{\mathfrak{d}_\tau}) = \check{v}_{\mathfrak{q}}(\kappa'(c)_{\mathfrak{q}\mathfrak{d}_\tau})$. Since the maps h and $c_{1-\tau}$ are isomorphisms, it is therefore enough to compute $(\iota_{\mathfrak{n},\mathfrak{q}} \circ h^{-1} \circ \check{\psi}^{\mathrm{fs}}_{\mathfrak{q}})(\kappa_{\mathfrak{n}})$ in terms of $(\iota_{\mathfrak{n},\mathfrak{q}} \circ c^{-1}_{1-\tau} \circ h^{-1} \circ \check{v}_{\mathfrak{q}})(\kappa_{\mathfrak{d}_\tau})$. Since Hypothesis 4.16 (ii) implies that the connecting homomorphism $\delta_{k(\mathfrak{n}),\mathfrak{q}}$ is injective, we may carry out this computation after applying $\delta_{k(\mathfrak{n}),\mathfrak{q}}$. Using Lemma 5.38 and 5.39 (i) and (ii) we are therefore reduced to proving that, in $H^1_f(k_{\mathfrak{q}}, \mathfrak{a}_i \mathcal{T}[\mathcal{G}_{k(\mathfrak{n})}])$, one has

$$\mathrm{Eul}_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}}) \cdot (D'_{\mathfrak{n}} c_{k(\mathfrak{n})}) = \sum\nolimits_{\tau \in \mathfrak{S}_{\mathfrak{q}}(\mathfrak{n})} \mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{l} \in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{l})}_{\tau(\mathfrak{l})}\big) \cdot (\sigma_{\mathfrak{q}} - 1) \cdot (D'_{\mathfrak{q}\mathfrak{d}_\tau} c_{k(\mathfrak{q}\mathfrak{d}_\tau)}). \quad (5.42)$$

To do this, we fix $\mathfrak{m} \in \mathcal{N}$ with $\mathfrak{m} \mid \mathfrak{n}$, and $\mathfrak{p} \in V(\mathfrak{m})$. If u is an element of $I_{\mathfrak{m}} + \mathfrak{a}_i \mathcal{R}[G_{\mathfrak{m}}]$, then u annihilates $(\sigma_{\mathfrak{p}} - 1)(D'_{\mathfrak{m}} c_{k(\mathfrak{m})})$. Indeed, this follows from the fact that u annihialates $\kappa'(c)_{\mathfrak{m}}$ (by Lemma 5.25) and Lemmas 5.38 and 5.39 (ii). For every $\mathfrak{l} \in V(\mathfrak{m})$ we have

$$\mathrm{Eul}_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}^{-1}) \equiv \sum\nolimits_{\mathfrak{p} \in V(\mathfrak{m}/\mathfrak{l})} x^{(\mathfrak{p})}_{\mathfrak{l}} (\sigma_{\mathfrak{p}} - 1) \mod (I_{\mathfrak{m}} + \mathfrak{a}_i \mathcal{R}[G_{\mathfrak{m}}])$$

and so it follows that

$$\begin{aligned} \mathrm{Eul}_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}) \cdot (D'_{\mathfrak{m}} c_{k(\mathfrak{m})}) &= \sum\nolimits_{\mathfrak{p} \in V(\mathfrak{m}/\mathfrak{l})} x^{(\mathfrak{p})}_{\mathfrak{l}} (\sigma_{\mathfrak{p}} - 1) D'_{\mathfrak{m}} c_{k(\mathfrak{m})} \\ &= -\sum\nolimits_{\mathfrak{p} \in V(\mathfrak{m}/\mathfrak{l})} x^{(\mathfrak{p})}_{\mathfrak{l}} \mathrm{Eul}_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}}^{-1}) \cdot D'_{\mathfrak{m}/\mathfrak{p}} c_{k(\mathfrak{n}/\mathfrak{p})}, \end{aligned} \quad (5.43)$$

where the second equality is by Lemma 5.39 (iii).
We can now use (5.43) (with $\mathfrak{m} = \mathfrak{n}$ and $\mathfrak{l} = \mathfrak{q}$) to express $-\mathrm{Eul}_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}}) \cdot (D'_{\mathfrak{n}} c_{k(\mathfrak{n})})$ as a sum of terms of the form $x^{(\mathfrak{p}_1)}_{\mathfrak{q}} \mathrm{Eul}_{\mathfrak{p}_1}(\mathrm{Frob}_{\mathfrak{p}_1}^{-1}) \cdot D'_{\mathfrak{m}/\mathfrak{p}_1} c_{k(\mathfrak{n}/\mathfrak{p}_1)}$. If $\mathfrak{p}_1 \neq \mathfrak{q}$, then we can apply (5.43) (with $\mathfrak{m} = \mathfrak{m}/\mathfrak{p}$ and $\mathfrak{l} = \mathfrak{p}$) to the corresponding summand in order to write it as a sum of terms $x^{(\mathfrak{p}_2)}_{\mathfrak{p}_1} \mathrm{Eul}_{\mathfrak{p}_2}(\mathrm{Frob}_{\mathfrak{p}_2}^{-1}) \cdot D'_{\mathfrak{m}/\mathfrak{p}_1\mathfrak{p}_2} c_{k(\mathfrak{n}/\mathfrak{p}_1\mathfrak{p}_2)}$. Continuing this process until $\mathfrak{p}_n = \mathfrak{q}$ for some n produces a cycle $\tau \coloneqq (\mathfrak{q}\, \mathfrak{p}_{n-1} \, \dots \, \mathfrak{p}_1) \in \mathfrak{S}_{\mathfrak{q}}(\mathfrak{n})$ such that $\mathfrak{d}_\tau \coloneqq \mathfrak{n} \cdot (\prod_{j=1}^n \mathfrak{p}_j)^{-1}$ and $\mathrm{sgn}(\tau) = (-1)^{n-1}$, and the resulting summand is

$$-\mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{l} \in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{l})}_{\tau(\mathfrak{l})}\big) \cdot (\sigma_{\mathfrak{q}} - 1)(D'_{\mathfrak{d}_\tau} c_{k(\mathfrak{d}_\tau)}).$$

This proves (5.42), thereby concluding the proof of the lemma. □

The next result now finally completes the proof of Theorem 5.27.

(5.44) Proposition. *Assume $\mathfrak{F}$ satisfies Hypothesis 4.1 and that $\mathcal{K}$ and $\mathcal{T}$ satisfy Hypothesis 4.16. Then, for every $c \in \mathrm{ES}^1(\mathfrak{F})$ and $\mathfrak{n} \in \mathcal{N}$, one has $\kappa(c)_{\mathfrak{n}} \in H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})$.*

Proof. If $\mathfrak{m} \mid \mathfrak{n}$, then $\kappa'(c)_{\mathfrak{m}}$ belongs to $H^1_{\tilde{\mathscr{F}}_{\mathfrak{m}}}(k, \mathcal{A}) \subseteq H^1_{\tilde{\mathscr{F}}_{\mathfrak{n}}}(k, \mathcal{A})$ and so the same is true for $\kappa(c)_{\mathfrak{n}}$. To prove that $\kappa(c)_{\mathfrak{n}}$ belongs to $H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})$ it is then enough, by the exact sequence (5.15), to prove $\check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}(\kappa(c)_{\mathfrak{n}}) = 0$ for all $\mathfrak{q} \in V(\mathfrak{n})$. To verify this, we write $U_{\mathfrak{q}}(\mathfrak{n}) \coloneqq \{\tau \in \mathfrak{S}(\mathfrak{n}) \mid \tau(\mathfrak{q}) = \mathfrak{q}\}$ for the stabiliser of $\mathfrak{q}$ in $\mathfrak{S}(\mathfrak{n})$ and note that every $\tau \in \mathfrak{S}(\mathfrak{n}) \setminus U$ can be written as $\tau = \sigma \circ \rho$ with $\sigma \in U_{\mathfrak{q}}(\mathfrak{n})$ and $\rho \in \mathfrak{S}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{d}_\sigma)$. Given this, we can rearrange the terms in the definition of $\kappa(c)_{\mathfrak{n}}$ as

$$\begin{aligned} \kappa(c)_{\mathfrak{n}} &= \sum\nolimits_{\tau \in \mathfrak{S}(\mathfrak{n})} \mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{q} \in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{q})}_{\tau(\mathfrak{q})}\big) \cdot \kappa'(c)_{\mathfrak{d}_\tau} \\ &= \Big(\sum\nolimits_{\tau \in U_{\mathfrak{q}}(\mathfrak{n})} \mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{q} \in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{q})}_{\tau(\mathfrak{q})}\big) \cdot \kappa'(c)_{\mathfrak{d}_\tau}\Big) \\ &\qquad + \Big(\sum\nolimits_{\sigma \in U_{\mathfrak{q}}(\mathfrak{n}), \rho \in \mathfrak{S}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{d}_\sigma)} \mathrm{sgn}(\sigma \circ \rho) \cdot \big(\prod\nolimits_{\mathfrak{q} \in V(\mathfrak{n}/\mathfrak{d}_{\sigma\rho})} x^{(\mathfrak{q})}_{(\sigma\rho)(\mathfrak{q})}\big) \cdot \kappa'(c)_{\mathfrak{d}_{\sigma\rho}}\Big) \\ &= \sum\nolimits_{\tau \in U_{\mathfrak{q}}(\mathfrak{n})} \mathrm{sgn}(\tau) \cdot \big(\prod\nolimits_{\mathfrak{q} \in V(\mathfrak{n}/\mathfrak{d}_\tau)} x^{(\mathfrak{q})}_{\tau(\mathfrak{q})}\big) \cdot \lambda_\tau \end{aligned}$$

with

$$\lambda_\tau := \kappa'(c)_{\mathfrak{d}_\tau} + \sum\nolimits_{\rho \in \mathfrak{S}_{\mathfrak{q}}(\mathfrak{n})} \mathrm{sgn}(\rho) \cdot \big(\prod\nolimits_{\mathfrak{q} \in V(\mathfrak{d}_\tau/\mathfrak{d}_\rho)} x^{(\mathfrak{q})}_{\rho(\mathfrak{q})}\big) \cdot \kappa'(c)_{\mathfrak{d}_\rho}.$$

By Lemma 5.41, we have $\check{\psi}^{\mathrm{fs}}_{\mathfrak{q}}(\lambda_\tau) = 0$ for every $\tau \in U_{\mathfrak{q}}(\mathfrak{n})$, so this shows the required vanishing of $\psi^{\mathrm{fs}}_{\mathfrak{q}}(\kappa(c)_{\mathfrak{n}})$. □

6. Kolyvagin systems II: controlling values at 1

Throughout this section we fix a natural number i and continue to use the notation in (5.1).

6.1. Tate–Shafarevich modules and relative core vertices

The notion of 'core vertex' plays a key role in the theory of Kolyvagin systems developed by Mazur and Rubin in [74]. In this subsection, we use the Cebotarev density theorem to prove the existence (under Hypotheses 4.14) of an appropriate analogue of core vertex in our theory.

6.1.1. Tate–Shafarevich modules and cohomological invariants

For every natural number $j \geq i$ and Mazur–Rubin structure $\mathcal{F}'$ on $\mathcal{A}$, we define $\mathbb{\Lambda}$-modules

$$Ш_{\mathcal{F}',j}(\mathcal{A}) = Ш_{\mathcal{F}'}(k, \mathcal{A}, \mathcal{Q}_j) := \ker\Big(H^1_{\mathcal{F}'}(k, \mathcal{A}) \to \prod\nolimits_{\mathfrak{q} \in \mathcal{Q}_j} H^1(k_{\mathfrak{q}}, \mathcal{A})\Big)$$

$$\mathfrak{X}_{\mathcal{F}',j}(\mathcal{A}) := H^1_{\mathcal{F}'}(k, \mathcal{A}) \cap H^1(\mathcal{G}_{k_j(\mathcal{T}_j)}, \mathcal{A}) = \ker\big(H^1_{\mathcal{F}'}(k, \mathcal{A}) \to H^1(k_j(\mathcal{T}_j), \mathcal{A})\big).$$

For a Nekovář structure $\mathscr{F}'$ on $\mathcal{A}$, the exact sequence (3.28) implies that, for $\mathfrak{q} \in \mathcal{Q}_j$, there exists a localisation map $H^1_{\mathscr{F}'}(k, \mathcal{A}) \to H^1(k_{\mathfrak{q}}, \mathcal{A})$. We can therefore similarly define a $\mathbb{\Lambda}$-module

$$Ш_{\mathscr{F}',j}(\mathcal{A}) = Ш_{\mathscr{F}'}(k, \mathcal{A}, \mathcal{Q}_j) := \ker\big(H^1_{\mathscr{F}'}(k, \mathcal{A}) \to \prod\nolimits_{\mathfrak{q} \in \mathcal{Q}_j} H^1(k_{\mathfrak{q}}, \mathcal{A})\big).$$

We thereby obtain increasing filtrations of Tate-Shafarevich modules

$$Ш_{\mathcal{F}',1}(\mathcal{A}) \subseteq Ш_{\mathcal{F}',2}(\mathcal{A}) \subseteq \cdots \subseteq H^1_{\mathcal{F}'}(k, \mathcal{A}) \quad \text{and} \quad Ш_{\mathscr{F}',1}(\mathcal{A}) \subseteq Ш_{\mathscr{F}',2}(\mathcal{A}) \subseteq \cdots \subseteq H^1_{\mathscr{F}'}(k, \mathcal{A}).$$

In the following result concerning these modules we write $\langle\tau\rangle$ for the subgroup of $\mathrm{Gal}(k_j(\mathcal{T}_j)/k_j)$ generated by τ.

(6.1) Lemma. *Assume Hypothesis 4.14 (i) and (ii). Then the following claims are valid.*

(i) $Ш_{\tilde{\mathcal{F}},j}(\mathcal{A}) \subseteq \mathfrak{X}_{\tilde{\mathcal{F}},j}(\mathcal{A})$, *with equality if* $H^1(\langle\tau\rangle, \mathcal{A})$ *vanishes.*

(ii) If $x \in \mathfrak{X}_{\tilde{\mathcal{F}},j}(\mathcal{A})$ *is such that* $\mathrm{loc}_v(x) = 0$ *for some* $v \in \mathcal{Q}_j$, *then* $x \in Ш_{\tilde{\mathcal{F}},j}(\mathcal{A})$.

(iii) There exists a canonical isomorphism of $\mathbb{\Lambda}$*-modules*

$$H^1_{\tilde{\mathscr{F}}}(k, \mathcal{A})/Ш_{\tilde{\mathscr{F}},j}(\mathcal{A}) \xrightarrow{\simeq} H^1_{\tilde{\mathcal{F}}}(k, \mathcal{A})/Ш_{\tilde{\mathcal{F}},j}(\mathcal{A}).$$

(iv) If Hypothesis 4.14 (iii) is also valid, then $Ш_{\overline{F}^*,j}(\overline{B})$ *vanishes.*

Proof. To prove the inclusion in (i), we need to show that any element ξ of $Ш_{\tilde{\mathcal{F}},j}(\mathcal{A})$ is trivial upon restriction to $G_{k_j(\mathcal{T}_j)}$. Since $G_{k_j(\mathcal{T}_j)}$ acts trivially on $\mathcal{A}$ (since $j \geq i$), the restriction $f := \mathrm{res}(\xi)$ of ξ to $G_{k_j(\mathcal{T}_j)}$ is a $\mathcal{G}_{k_j(\mathcal{T}_j)}$-equivariant homomorphism $f \colon G_{k_j(\mathcal{T}_j)} \to \mathcal{A}$. In particular, f cuts out a finite extension K of $k_j(\mathcal{T}_j)$ that is Galois over k.
Now let σ be an element in $\mathrm{Gal}(K/k_j(\mathcal{T}_j))$. Since $\tau\sigma$ agrees with τ when restricted to $k_j(\mathcal{T}_j)$, Cebotarev's density theorem allows us to choose places v_1 and v_2 in $\mathcal{Q}_j$ such that Frob_{v_1} and Frob_{v_2} are equal to $\tau\sigma$ and τ, respectively, in $\mathcal{G}_K$. By assumption, ξ is trivial when restricted to the decomposition groups of v_1 and v_2, so there are elements $a_1, a_2 \in \mathcal{A}$ with $\xi(\rho) = (\rho - 1) \cdot a_n$

if ρ belongs to the decomposition group of v_n. It follows that

$$\begin{aligned}\xi(\mathrm{Frob}_{v_2}^{-1}\mathrm{Frob}_{v_1}) &= \mathrm{Frob}_{v_2}^{-1}\cdot\xi(\mathrm{Frob}_{v_1}) + \xi(\mathrm{Frob}_{v_2}^{-1})\\ &= \mathrm{Frob}_{v_2}^{-1}\mathrm{Frob}_{v_1}a_1 - \mathrm{Frob}_{v_2}^{-1}a_1 + \mathrm{Frob}_{v_2}^{-1}a_2 - a_2\\ &= a_1 - \tau^{-1}a_1 + \tau^{-1}a_2 - a_2\\ &= (\tau^{-1}-1)\cdot(a_2 - a_1).\end{aligned}$$

Now, $\mathrm{Frob}_{v_2}^{-1}\mathrm{Frob}_{v_1} \in G_{k_j(\mathcal{T}_j)}$ and so $\xi(\mathrm{Frob}_{v_2}^{-1}\mathrm{Frob}_{v_1}) = f(\mathrm{Frob}_{v_2}^{-1}\mathrm{Frob}_{v_1}) = f(\tau^{-1}\tau\sigma) = f(\sigma)$. The above calculation therefore implies that $\mathrm{im}(f)$ is contained in $(\tau-1)\mathcal{A}$. Hypothesis 4.14 (i) and (ii) then combines with Lemma 6.2 below (applied with $L = k_j(\mathcal{T}_j)$) to imply that f is trivial, as required to prove the inclusion in (i).
Next we note that, if $H^1(\langle\tau\rangle,\mathcal{A})$ vanishes, then the inflation-restriction sequence shows every element x of $\mathfrak{X}_{\mathcal{F},j}(\mathcal{A})$ is the inflation of an element of $H^1(k_j(\mathcal{T}_j)^{\langle\tau\rangle}/k,\mathcal{A})$. Since every place in $\mathcal{Q}_j$ splits completely in $k_j(\mathcal{T}_j)^{\langle\tau\rangle}$ by definition, it follows that $\mathrm{loc}_v(x)$ must be trivial for every $v\in\mathcal{Q}_j$. This shows that x belongs to $\text{Ш}_{\mathcal{F},j}(\mathcal{A})$, as required to complete the proof of (i).
If x is a cohomology class in $\mathfrak{X}_{\mathcal{F},j}(\mathcal{A})$, then by definition x is trivial when restricted to $G_{k_j(\mathcal{T}_j)}$ and, in particular, x is unramified at every place that is unramified in $k_j(\mathcal{T}_j)/k$. In particular, x is unramified at every place in $\mathcal{Q}_j$ and so the restriction $\mathrm{loc}_v(x)$ belongs to $H^1_f(k_v,\mathcal{A})$. From the isomorphism

$$H^1_f(k_v,\mathcal{A}) \xrightarrow{\simeq} \mathcal{A}/(\tau-1)\mathcal{A}, \quad [y]\mapsto y(\mathrm{Frob}_v) \mod (\tau-1)$$

for any $v\in\mathcal{Q}_j$, we see that $\mathrm{res}_v(x)$ vanishes if and only if the element

$$x(\mathrm{Frob}_v) = x(\tau\tau^{-1}\mathrm{Frob}_v) = x(\tau) + \tau^{-1}x(\tau^{-1}\mathrm{Frob}_v) = x(\tau)$$

belongs to $(\tau-1)\mathcal{A}$. Since the latter condition does not depend on v, we see that x belongs to $\text{Ш}_{\mathcal{F},j}(\mathcal{A})$ if $\mathrm{res}_v(x)$ vanishes for any given $v\in\mathcal{Q}_j$, as claimed in (ii).
For $\mathfrak{q}\in S(\tilde{\mathscr{F}})$, the natural localisation map $H^1_{\tilde{\mathscr{F}}}(k,\mathcal{A})\to H^1(k_{\mathfrak{q}},\mathcal{A})$ factors through the map $H^1_{\tilde{\mathscr{F}}}(k,\mathcal{A})\to H^1(\mathcal{O}_{k,S(\tilde{\mathscr{F}})},\mathcal{A})$ induced by the triangle (3.10) and the exact sequence (3.28) implies that the latter map has image $H^1_{\tilde{\mathcal{F}}}(k,\mathcal{A})$. Claim (iii) follows directly from these facts.
To prove (iv), we note Hypothesis 4.14 (iii) implies $H^1(\mathcal{G}_{k_j(\mathcal{T}_j)},\overline{B})$, and hence also its subgroup $\mathfrak{X}_{\overline{F}^*(1),j}(\overline{B})$, vanishes (cf. Remark 4.15 (i) and (ii)). The vanishing of $\text{Ш}_{\overline{F}^*(1),j}(\overline{B})$ now follows from the argument proving the inclusion in (i), after replacing $\tilde{\mathcal{F}}$ and $\mathcal{A}$ by $\overline{F}^*$ and $\overline{B}$. □

(6.2) Lemma. *Assume Hypotheses 4.14 (i) and (ii). If L is a finite Galois extension of k containing $k_i(\mathcal{A})$, then the following natural map is injective*

$$H^1(L,\mathcal{A})^{\mathcal{G}_L} = \mathrm{Hom}_{\mathcal{G}_L}(G_L,\mathcal{A}) \to \mathrm{Hom}(G_L,\mathcal{A}/(\tau-1)\mathcal{A}).$$

Proof. Let $x\colon G_L\to\mathcal{A}$ be a non-trivial homomorphism in $\mathrm{Hom}_{\mathcal{G}_L}(G_L,\mathcal{A})$ that belongs to the kernel of the above map. That is, $\mathrm{im}(x)$ is contained in $(\tau-1)\mathcal{A}$. Then, since x is non-trivial, Lemma 2.9 implies the existence of a non-zero element λ of Λ such that $\lambda\cdot x$ is a non-zero element of the $\mathcal{M}_i$-torsion submodule of $\mathrm{Hom}_{\mathcal{G}_L}(G_L,\mathcal{A})$. In particular $\mathrm{im}(\lambda x) = \lambda\cdot\mathrm{im}(x)$ is contained in $\mathcal{A}[\mathcal{M}_i]$. In addition, following Remark 2.10, there exists an isomorphism $\mathcal{A}\otimes_{\mathbb{\Lambda}}\mathbb{K}\cong\mathcal{A}[\mathcal{M}_i]$ and so Hypothesis 4.14 (i) implies $\mathcal{A}[\mathcal{M}_i]$ has no proper non-trivial G_k-stable submodules. If $\lambda\cdot\mathrm{im}(x)$ is non-zero, it must therefore span $\mathcal{A}[\mathcal{M}_i]$ over $\mathbb{\Lambda}$. Since $\mathrm{im}(x)$, and hence $\lambda\cdot\mathrm{im}(x)$, is contained in the $\mathbb{\Lambda}$-module $(\tau-1)\mathcal{A}$, it follows that $(\tau-1)\mathcal{A}$ contains $\mathcal{A}[\mathcal{M}_i]$. We therefore obtain a surjective map of Λ-modules $\theta\colon\mathcal{A}/(\mathcal{A}[\mathcal{M}_i])\twoheadrightarrow\mathcal{A}/(1-\tau)\mathcal{A}$. If $\mathcal{A}=\mathcal{A}\otimes_{\mathbb{\Lambda}}\mathbb{K}$, then the domain of θ vanishes, whilst its codomain is isomorphic to $\mathbb{K}$ and this is a contradiction. On the other hand, if $i\geq 1$, then the codomain of θ is isomorphic to $\mathbb{\Lambda}$, whilst Lemma 2.9 implies that every element of its domain is annihilated by a non-zero element of $\mathbb{\Lambda}$. The surjectivity of θ therefore implies that $1\in\mathbb{\Lambda}$ is annihilated by a non-zero element of $\mathbb{\Lambda}$ and this is a contradiction. In all cases, therefore, the map x must be trivial, as required. □

If $j \geq i$, then for any pairwise coprime moduli $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{n}$ in $\mathcal{N}_j$, one has

$$Ш_{\overline{F},j}(\overline{A}) \subseteq H^1_{\overline{F}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})}(k,\overline{A}) \text{ and } Ш_{\overline{F}^*,j}(\overline{B}) \subseteq H^1_{(\overline{F}^*)^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{n})}(k,\overline{B}).$$

In particular if, for $\mathfrak{m} \in \mathcal{N}_j$, we set

$$\lambda_{\overline{F}}(\mathfrak{m},j) := \dim_{\Bbbk}\big(H^1_{\overline{F}(\mathfrak{m})}(k,\overline{A})/Ш_{\overline{F},j}(\overline{A})\big)$$
$$\lambda^*_{\overline{F}}(\mathfrak{m},j) := \dim_{\Bbbk}\big(H^1_{\overline{F}^*(\mathfrak{m})}(k,\overline{B})/Ш_{\overline{F}^*,j}(\overline{B})\big),$$

then the integer

$$\chi(\overline{F},j) := \lambda_{\overline{F}}(\mathfrak{m},j) - \lambda^*_{\overline{F}}(\mathfrak{m},j)$$

provides an appropriate analogue in our theory of the cohomological invariants for Mazur–Rubin structures introduced in [74, §4.1]. The basic properties of these integers are as follows.

(6.3) Lemma. *Fix $j \geq i$. Then $\chi(\overline{F},j)$ is independent of $\mathfrak{m}$. Further, for all pairwise coprime moduli $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{n}$ in $\mathcal{N}_j$, the following claims are valid.*

(i) $\chi(\overline{F}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n}),j) = \chi(\overline{F},j) + \nu(\mathfrak{b}) - \nu(\mathfrak{a})$.

(ii) $\chi(\overline{F},j) = \chi(\overline{F}(\mathfrak{n}),j)$.

(iii) If Hypothesis 4.14 (i), (ii) and (iii) are satisfied, then $\lambda^*_{\overline{F}}(\mathfrak{n},j) = \dim_{\Bbbk}(H^1_{\overline{F}^*(\mathfrak{n})}(k,\overline{B}))$.

Proof. Both the independence of $\chi(\overline{F},j)$ from $\mathfrak{m}$ and the equality in (i) follow from the argument of [93, Cor. 3.21]. Specifically, since

$$\lambda_{\overline{F}^{\mathfrak{b}}_{\mathfrak{a}}}(\mathfrak{n},j) = \dim_{\Bbbk}(H^1_{\overline{F}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})}(k,\overline{A})) - \dim_{\Bbbk}(Ш_{\overline{F},j}(\overline{A}))$$
$$\lambda^*_{\overline{F}^{\mathfrak{b}}_{\mathfrak{a}}}(\mathfrak{n},j) = \dim_{\Bbbk}(H^1_{(\overline{F}^*)^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})}(k,\overline{B})) - \dim_{\Bbbk}(Ш_{\overline{F}^*,j}(\overline{B})),$$

one need only compare dimensions by using appropriate cases of the global duality exact sequences (5.14)–(5.17) with $\mathbb{A}, \mathcal{A}$ and both Φ and $\tilde{\mathcal{F}}$ taken to be $\Bbbk, \overline{A}$ and $\overline{F}$ respectively.
Claim (ii) then follows directly from (i) in the case $\mathfrak{a} = \mathfrak{b} = 1$, and (iii) is an immediate consequence of Lemma 6.1 (iv). □

(6.4) Remark. Lemma 6.3 (iii) implies that, if Hypothesis 4.14 is satisfied, then the non-negative integer $\lambda^*_{\overline{F}}(\mathfrak{n},j)$ is independent of j. In such cases, we abbreviate $\lambda^*_{\overline{F}}(\mathfrak{n},j)$ to $\lambda^*_{\overline{F}}(\mathfrak{n})$.

6.1.2. The Cebotarev density theorem

In the sequel, for $j \geq i$ we write $Ш_j(\mathcal{A})$ for the subset of $H^1(k,\mathcal{A})$ comprising all classes that are locally trivial at every place in $\mathcal{Q}_j$ and $\mathfrak{X}_j(\mathcal{A})$ for $H^1(\mathcal{G}_{k_j(\mathcal{T}_j)},\mathcal{A})$, regarded as a submodule of $H^1(k,\mathcal{A})$ via the inflation map.
Then the following consequence of the Cebotarev density theorem constitutes a refined version of both [74, Prop. 3.6.1] and [26, Lem. 3.9].

(6.5) Proposition. *Assume to be given data of one of the following forms:*

(i) Hypotheses 4.14 (i), (ii) and (ii) are satisfied; $Z \in \{\mathcal{A}, \overline{\mathcal{A}}, A, \overline{A}\}$ and $D = \{c_1, c_1^*\}$ where, for some $j \geq i$, one has*

$$c_1 \in (H^1(k,Z)/Ш_j(Z)) \setminus \{0\} \text{ and } c_1^* \in (H^1(k,Z^*(1))/Ш_j(Z^*(1))) \setminus \{0\}.$$

(ii) Hypotheses 4.14 (i), (ii), (ii) and (iv) are satisfied; $Z_1 \in \{A, \overline{A}\}$, $Z_2 \in \{\mathcal{A}, \overline{\mathcal{A}}, A, \overline{A}\}$ and $D = \{c_1, c_2, c_1^*, c_2^*\}$ where, for some $j \geq n$ and both $s = 1$ and $s = 2$, one has*

$$c_s \in (H^1(k,Z_s)/Ш_j(Z_s)) \setminus \{0\} \text{ and } c_s^* \in (H^1(k,Z_s^*(1))/Ш_j(Z_s^*(1))) \setminus \{0\}.$$

Then there exists a subset $\mathcal{S} \subseteq \mathcal{Q}_j$ of positive density with $\psi^{\mathrm{fs}}_{\mathfrak{q}}(c) \neq 0$ for all $c \in D$ and $\mathfrak{q} \in \mathcal{S}$.

Proof. Note, at the outset, that we may assume that c_s does not belong to $\mathfrak{X}_j(Z_s)$ for $s \in \{1,2\}$. Indeed, if $c_s \in \mathfrak{X}_j(Z_s)/Ш_j(Z_s)$ is nonzero, then Lemma 6.1 (iii) implies that $\mathrm{loc}_v(c_s) \neq 0$ for all $v \in \mathcal{Q}_j$ and so the claim is trivial for c_s. Similarly, we may assume that c_s^* does not belong to $\mathfrak{X}_j(Z_s^*(1))$ for all $s \in \{1,2\}$.

Next we observe that, for $\mathfrak{q} \in \mathcal{Q}_j$, there exists a unit $u_\mathfrak{q}$ (in $\mathbb{\Lambda}$, Λ, $\mathbb{K}$ or $\mathbb{k}$ as appropriate, and only depending on $\mathfrak{q}$) with

$$\psi_\mathfrak{q}^{\mathrm{fs}}(x) = u_\mathfrak{q} \cdot x(\mathrm{Frob}_\mathfrak{q}) \in Z/(\mathrm{Frob}_\mathfrak{q} - 1) \quad \text{for all} \quad x \in H^1(k, Z),$$

and similarly for $Z^*(1)$. Writing $f_\mathfrak{q} \in H^1(k,Z)^*$ and $f_\mathfrak{q}^* \in H^1(k, Z^*(1))^*$ for the maps defined by $x \mapsto x(\mathrm{Frob}_\mathfrak{q})$, it therefore suffices to prove that there exists a subset $\mathcal{S} \subseteq \mathcal{Q}_j$ of positive density with the property that $f_\mathfrak{q}(c_1) \neq 0$ and $f_\mathfrak{q}^*(c_1^*)$ for every $\mathfrak{q} \in \mathcal{S}$ in order to establish the claim in case (i). To do this, we set $\Delta \coloneqq \mathrm{Gal}(k_j(\mathcal{T}_j)/k)$ and consider the composite map

$$\phi\colon H^1(k,Z) \xrightarrow{\mathrm{Res}} H^1(k_j(\mathcal{T}_j), Z)^\Delta = \mathrm{Hom}_\Delta(G_{k_j(\mathcal{T}_j)}, Z) \to \mathrm{Hom}(G_{k_j(\mathcal{T}_j)}, Z/(\tau - 1)Z)$$

where the first arrow denotes restriction and the second is induced by $Z \to Z/(\tau-1)Z$.

By assumption, c_1 does not belong to $\mathfrak{X}_j(M)$, hence not to the kernel of the first arrow. Since the second arrow is injective by Lemma 6.2, we therefore have that $\phi(c_1) \neq 0$ and so $\ker \phi(c_1)$ is a proper subgroup of $G_{k_j(\mathcal{T}_j)}$. Consider the subset $H_1 \coloneqq \phi(c_1)^{-1}(-c_1(\tau))$ of $G_{k_j(\mathcal{T}_j)}$. If $g_1, g_2 \in H_1$, then one has that $g_1 g_2^{-1}$ belongs to the kernel of $\phi(c_1)$ (since $\phi(c_1)$ is a homomorphism). This shows that H_1 is a coset of $\ker(\phi(c_1))$.

In exactly the same way we set $H_1^* \coloneqq \phi'(c_1^*)^{-1}(-c_1^*(\tau))$, with ϕ' given by the composite map

$$\begin{aligned}\phi'\colon H^1(k, Z^*(1)) &\xrightarrow{\mathrm{Res}^*} H^1(k_j(\mathcal{T}_j), Z^*(1))^\Delta \\ &= \mathrm{Hom}_\Delta(G_{k_j(\mathcal{T}_j)}, Z) \to \mathrm{Hom}_\Delta(G_{k_j(\mathcal{T}_j)}, Z^*(1)/(\tau-1)).\end{aligned}$$

One then has that H_1^* is coset of the proper subgroup $\ker(\phi'(c_1^*))$ of $G_{k_j(\mathcal{T}_j)}$.

It now follows from the general result of Lemma 6.7 below that $H_1 \cup H_1^*$ is a proper subset of $G_{k_j(\mathcal{T}_j)}$. (Here we are using that, if $\tau = \mathrm{id}$, then $H_1 = \ker(\phi(c_1))$ and $H_1^* = \ker(\phi'(c_1^*))$ are subgroups of $G_{k_j(\mathcal{T}_j)}$).

Fix an element $\gamma \in G_{k_j(\mathcal{T}_j)} \setminus (H_1 \cup H_1^*)$ and write $L \coloneqq L_1 L_1^*$ for the finite extension of $k_j(\mathcal{T}_j)$ defined as the composite of the extensions L_1 and L_1^* that are cut out by the kernels of $\mathrm{Res}(c_1)$ and $\mathrm{Res}^*(c_1^*)$, respectively. Let $\mathcal{S} \subseteq \mathcal{Q}_j$ be the subset of primes that are both coprime to $\mathfrak{n}$ and such that the restriction of $\mathrm{Frob}_\mathfrak{q}$ to L agrees with $\tau\gamma$. By construction, for every such $\mathfrak{q} \in \mathcal{S}$ one then has $\tau^{-1}\mathrm{Frob}_\mathfrak{q} \in G_{k_j(\mathcal{T}_j)}$ and so, since the cocycle relation implies $c_s(\tau\tau^{-1}\mathrm{Frob}_\mathfrak{q}) = c_s(\tau) + \tau c_s(\tau^{-1}\mathrm{Frob}_\mathfrak{q})$, we can compute

$$\begin{aligned} c_s(\mathrm{Frob}_\mathfrak{q}) = c_s(\tau\tau^{-1}\mathrm{Frob}_\mathfrak{q}) &\equiv c_s(\tau) + \phi(c_i)(\tau^{-1}\mathrm{Frob}_\mathfrak{q}) \mod (\tau-1)\mathcal{A} \\ &\equiv c_s(\tau) + \phi(c_s)(\gamma) \mod (\tau-1)\mathcal{A} \\ &\not\equiv 0 \mod (\tau-1)\mathcal{A}. \end{aligned} \tag{6.6}$$

Here the second congruence is true since $\tau^{-1}\mathrm{Frob}_\mathfrak{q} \in \gamma\ker(\phi(c_s))$ and the final assertion as $\gamma \notin H_s$. The non-vanishing of each element $c_s(\mathrm{Frob}_\mathfrak{q})$ implies that $\mathcal{S}$ has all of the required properties, and hence concludes the proof of the claim in case (i).

In case (ii) it is convenient to argue separately for the cases $p > 3$ and $p \in \{2,3\}$.

If $p > 3$, then for $a \in \{1,2\}$ we set $H_a \coloneqq \phi(c_a)^{-1}(-c_2(\tau))$ and $H_a^* \coloneqq \phi'(c_a^*)^{-1}(-c_a^*(\tau))$ and take L_a and L_a^* to be the fields that are respectively cut out by the kernels of $\mathrm{Res}(c_a)$ and $\mathrm{Res}^*(c_a^*)$. Then, since a simple counting argument (using $p > 3$) shows that $X \coloneqq H_1 \cup H_2 \cup H_1^* \cup H_2^*$ is not equal to $G_{k_j(\mathcal{T}_j)}$, we can therefore again fix $\gamma \in G_{k_j(\mathcal{T}_j)} \setminus X$ and take $\mathcal{S}$ to be the subset of primes that are coprime to $\mathfrak{n}$ and such that the restriction of $\mathrm{Frob}_\mathfrak{q}$ to the compositum $L_1 L_2 L_1^* L_2^*$ agrees with $\tau\gamma$.

In the rest of the argument, we thus assume $p \in \{2,3\}$. In this case, we can first use Lemma 2.9 to choose elements y_1, y_2, y_1^*, y_2^* of R such that each $y_s^\bullet \mathrm{Res}^\bullet(c_s)$ is a nonzero element of

$$\mathrm{Hom}_\Delta(G_{k_j(\mathcal{T}_j)}, Z_s)[\mathfrak{a}_s] = \mathrm{Hom}_\Delta(G_{k_j(\mathcal{T}_j)}, Z_s[\mathfrak{a}_s]) \cong \mathrm{Hom}_\Delta(G_{k_j(\mathcal{T}_j)}, \overline{Z_s}).$$

(Here the symbol $\bullet$ is either $*$ or dropped, $\mathfrak{a}_s$ is either $\mathcal{M}_i$ or M_i depending on if Z_s is a $\mathbb{\Lambda}$ or Λ-module, $\overline{Z_s} := Z_s/\mathfrak{a}_s Z_s$, and the isomorphism is induced by a choice of isomorphism $\mathbb{k} \cong \Lambda[\mathfrak{a}_i]$ resp. $\mathbb{K} \cong \mathbb{\Lambda}$.) It is then enough to find a subset $\mathcal{S} \subseteq \mathcal{Q}_j$ of positive density such that $\mathrm{loc}_v(y_1 c_1), \mathrm{loc}_v(y_2 c_2), \mathrm{loc}_v(y_1^* c_1^*), \mathrm{loc}_v(y_2^* c_2^*)$ are all nonzero for every $v \in \mathcal{S}$.
Define finite extensions L_1, L_2, L_1^*, L_2^* of $k_j(\mathcal{T}_j)$ as the fields cut out by the kernels of $y_1\mathrm{Res}(c_1)$, $y_2\mathrm{Res}(c_2)$, $y_1^*\mathrm{Res}^*(c_1^*)$, $y_2^*\mathrm{Res}^*(c_2^*)$. We then set $L' := L_1L_2 \cap L_1^*L_2^*$ and claim that

$$L' = k_j(\mathcal{T}_j).$$

To show this, we note $\mathrm{Gal}(L'/k_j(\mathcal{T}_j))$ is a quotient of the subgroup $\mathrm{Gal}(L_1L_2/k_j(\mathcal{T}_j))$ of $\mathrm{Gal}(L_1/k_j(\mathcal{T}_j)) \times \mathrm{Gal}(L_2/k_j(\mathcal{T}_j))$, which we can identify with a submodule of $\overline{A} \oplus \overline{Z_2}$ via $(y_1\mathrm{Res}(c_1), y_2\mathrm{Res}(c_2))$. This shows that $\mathrm{Gal}(L'/k_j(\mathcal{T}_j))$ is isomorphic to a $\mathbb{Z}_p[G_k]$-subquotient of $\overline{T} \oplus \overline{Z_2}$. Similarly, $(y_1^*\mathrm{Res}(c_1^*), y_2^*\mathrm{Res}(c_2^*))$ induces an isomorphism between $\mathrm{Gal}(L'/k_j(\mathcal{T}_j))$ and a $\mathbb{Z}_p[G_k]$-subquotient of $\overline{B} \oplus \overline{Z_2}^*(1)$. In order to deduce $\mathrm{Gal}(L'/k_j(\mathcal{T}_j))$ is trivial, and hence that $L' = k_j(\mathcal{T}_j)$, it is thus enough to show Hypothesis 4.14 (iv) implies that the $\mathbb{Z}_p[G_k]$-modules $\overline{A} \oplus \overline{Z_2}$ and $\overline{B} \oplus \overline{Z_2}^*(1)$ can have no non-zero isomorphic subquotients. In addition, since $\mathbb{Z}_p[G_k]$ acts on these modules via a finite (and hence Artinian) quotient ring, the modules have composition series and so, by the Jordan–Hölder Theorem, it is enough to show that they have no isomorphic composition factors (as $\mathbb{Z}_p[G_k]$-modules). Now every composition factor of $\overline{A} \oplus \overline{Z_2}$ is isomorphic to a composition factor of either $\overline{A}$ or $\overline{Z_2} \in \{\overline{A}, \overline{\mathcal{A}}\}$ and hence to a composition factor of $\overline{A} \oplus \overline{\mathcal{A}}$. Similarly, every composition factor of $B \oplus \overline{Z_2}^*(1)$ is isomorphic to a composition factor of $B \oplus \overline{\mathcal{A}}^*(1)$. It is thus enough to show that no composition factor of $\overline{A} \oplus \overline{\mathcal{A}}$ is isomorphic to a composition factor of $\overline{B} \oplus \overline{\mathcal{A}}^*(1)$ and this follows directly from Hypothesis 4.14 (iv).
We set $H_a^\bullet := \phi(y_a^\bullet c_a^\bullet)^{-1}(-y_a^\bullet c_a^\bullet)$ for $a \in \{1, 2\}$. We then use the same argument as in case (i) to choose elements $\gamma, \gamma^* \in G_{k_j(\mathcal{T}_j)}$ such that $\gamma \notin H_1 \cup H_2$ and $\gamma^* \notin H_1^* \cup H_2^*$, Since $L' = k_j(\mathcal{T}_j)$, we can then find an element γ' of $G_{k_j(\mathcal{T}_j)}$ such that the restrictions of γ' to L_1L_2 and $L_1^*L_2^*$ are equal to γ and γ^* respectively. The subset $\mathcal{S}$ of $\mathcal{Q}_j$ comprising all primes $\mathfrak{q}$ for which $\mathrm{Frob}_\mathfrak{q}$ is conjugate to $\tau\gamma'$ in $\mathrm{Gal}(L_1L_2L_1^*L_2^*/k)$ is then easily seen to have the required properties. □

The following general observation was used in the above argument.

(6.7) Lemma. *Let G be an infinite group, and $U_1, U_2 \subsetneq G$ two normal subgroups of finite index. Let V_1 be a coset of U_1 in G, and V_2 a coset of U_2 in G. Then $G \neq V_1 \cup V_2$ unless $U_1 = U_2$ is a subgroup of index 2 and V_1 and V_2 are the two cosets in G/U_1.*

Proof. Suppose $G = V_1 \cup V_2$. By assumption $U_1, U_2 \subsetneq G$, so also must have $V_1, V_2, \subsetneq G$. We can therefore find elements $x, y \in G$ with $x \notin V_1$ and $y \notin V_2$. It follows that $x \in V_2$ and $y \in V_1$. If $z := xy \in V_1$, then $zy^{-1} = x \in U_1$. Similarly, if $z \in V_2$, then $x^{-1}z = y \in U_2$.
If $(G : U_1) > 2$ and $(G : U_2) > 2$, then we may choose $x \notin U_1 \cup V_1$ and $y \notin U_2 \cup V_2$, which contradicts the previous conclusion. Without loss of generality we may therefore assume that $(G : U_1) = 2$. In particular, we have $G = V_1 \cup wU_1$ with w an element of G with $w \notin V_1$. In this case we therefore have $V_1' := wU_1 \subseteq V_2$.
Now let u_1 be an element of U_1. Then u must belong to V_1 or V_1'. In the first case $V_1 \cap U_1 \neq \varnothing$ and so $V_1 = U_1$. In the second case $V_1' \cap U_1 \neq \varnothing$ and so $V_1' = U_1$. Similarly an element u_2 of U_2 must belong to V_1 or V_1'. If it belongs to $V_1' \subseteq V_2$, then $U_2 \cap V_2 \neq \varnothing$ and so $V_2 = U_2$. Since u_2 was arbitrary, this is the case if *any* element of U_2 belongs to V_1'. Otherwise we therefore must have $U_2 \cap V_1' = \varnothing$, and hence $U_2 \subseteq V_1$.
It is thus enough to consider separately the following four cases:

- $V_1 = U_1$ and $V_2 = U_2$. In this case one has $G = U_1 \cup U_2$ and so an elementary argument in group theory (in fact, the same as at the beginning of this proof) implies $G = U_1$ or $G = U_2$, which contradicts the hypothesis that U_1 and U_2 are strict subgroups of G.
- $V_1 = U_1$ and $U_2 \subseteq V_1$. In this case $U_2 \subseteq V_1 = U_1$ and so the argument in the beginning of the proof implies U_2 has index 2. Since U_1 also has index 2, it follows that $U_1 = U_2$.

- $V_1' = U_1$ and $V_2 = U_2$. In this case $U_1 = V_1' \subseteq V_2 = U_2$. Since $(G : U_1) = 2$ and $U_2 \neq G$, it follows that $U_1 = U_2$.
- $V_1' = U_1$ and $U_2 \subseteq V_1$. In this case $U_1 \cap U_2 \subseteq V_1' \cap V_1 = \varnothing$. However, since G is infinite, and both U_1 and U_2 have finite index in G, this is impossible.

□

6.1.3. Relative core vertices

The following is an analogue in our theory of the notion of core vertex from [74, Def. 4.1.8].

(6.8) Definition. *A 'relative core vertex' for $\tilde{\mathcal{F}}$ on $\mathcal{A}$ is a modulus $\mathfrak{n}$ in $\mathcal{N}$ for which the group $H^1_{\tilde{F}^*(\mathfrak{n})}(k, B)$ vanishes.*

(6.9) Remark. This notion is relative to the given ring homomorphism $\mathbb{A} \to \Lambda$, and is a weakening of the notion of core vertex used by Mazur and Rubin. To justify the latter observation, let $\mathfrak{n}$ is a core vertex for $\tilde{\mathcal{F}}$ in the sense of [76, Def. 4.1.8]. Then $H^1_{\tilde{\mathcal{F}}^*(\mathfrak{n})}(k, \mathcal{B})$ vanishes and sp, from [26, Cor. 3.8], it follows that $H^1_{(\tilde{\mathcal{F}}_A)^*(\mathfrak{n})}(k, B)$ also vanishes, where $\tilde{\mathcal{F}}_A$ is the Mazur–Rubin structure on A induced by $\tilde{\mathcal{F}}$ (as in Example 3.23 (iv)). In particular, since $\tilde{\mathcal{F}}_A \leq \tilde{F}$ (cf. Remark 3.24 (ii)), and hence also $\tilde{F}^*(\mathfrak{n}) \leq (\tilde{\mathcal{F}}_A)^*(\mathfrak{n})$, the group $H^1_{\tilde{F}^*(\mathfrak{n})}(k, B)$ vanishes. It follows that any core vertex for $\tilde{\mathcal{F}}$ in the sense of Mazur and Rubin is a relative core vertex for $\tilde{\mathcal{F}}$ in the above sense. However, the converse may not be true. In fact, even though our next result shows that relative core vertices for $\tilde{\mathcal{F}}$ always exist under Hypotheses 4.14, it is possible that $H^1_{\tilde{\mathcal{F}}^*(\mathfrak{n})}(k, \mathcal{B})$ is nonzero for every $\mathfrak{n} \in \mathcal{N}$. Fortunately, however, if $\mathfrak{n}$ is a relative core vertex for $\tilde{\mathcal{F}}$, then in all cases one can usefully 'bound' the complexity of the $\mathbb{A}$-module $H^1_{\tilde{\mathcal{F}}^*(\mathfrak{n})}(k, \mathcal{B})$ (see Remark 6.15 below).

(6.10) Remark. Lemma 2.9 implies $H^1_{\tilde{F}^*(\mathfrak{n})}(k, B)$ vanishes if and only if $H^1_{\tilde{F}^*(\mathfrak{n})}(k, B)[M_i]$ vanishes, and Lemma 3.37 implies the latter module is isomorphic to $H^1_{\overline{F}^*(\mathfrak{n})}(k, \overline{B})$. Hence, if Hypothesis 4.14 (i), (ii) and (iii) are satisfied, then Lemma 6.3 (iii) implies that $\mathfrak{n}$ is a relative core vertex for $\tilde{\mathcal{F}}$ if and only if the (non-negative) integer $\lambda^*_{\overline{F}}(\mathfrak{n})$ defined in Remark 6.4 is equal to 0, or equivalently one has

$$\chi(\overline{F}, j) = \dim_{\mathbb{k}}(H^1_{\overline{F}(\mathfrak{n})}(k, \overline{A})) - \dim_{\mathbb{k}}(\text{Ш}_{\overline{F},j}(\overline{A})) \tag{6.11}$$

for each $j \geq i$.

The next result guarantees the existence of relative core vertices with certain additional properties that will be essential in our later arguments.

(6.12) Lemma. *Assume Hypotheses 4.14. Then for every $j \in \mathbb{N}$ with $j \geq i$, there exists a relative core vertex $\mathfrak{n}$ for $\tilde{\mathcal{F}}$ that belongs to $\mathcal{N}_j (\subseteq \mathcal{N})$ and is such that $\nu(\mathfrak{n})$ is equal to the integer $\lambda^*_{\overline{F}}(1)$ defined in Remark 6.4.*

Proof. Set $s \coloneqq \lambda^*_{\overline{F}}(1) = \dim_{\mathbb{k}}(H^1_{\overline{F}^*}(k, \overline{B}))$. Then we shall use an induction on s to construct a modulus $\mathfrak{n}$ in $\mathcal{N}_j$ such that $\nu(\mathfrak{n}) = \lambda^*_{\overline{F}}(1)$ and $H^1_{\overline{F}^*(\mathfrak{n})}(k, \overline{B})$ vanishes. For such an $\mathfrak{n}$, Lemma 3.37 implies $H^1_{\tilde{F}^*(\mathfrak{n})}(k, B)[M_i]$ vanishes and hence that $H^1_{\tilde{F}^*(\mathfrak{n})}(k, B)$ vanishes as a consequence of Lemma 2.9. It follows that any such modulus $\mathfrak{n}$ is a relative core vertex for $\tilde{\mathcal{F}}$ of the required form.
If, firstly, $s = 0$, then the claim is clearly satisfied (with $\mathfrak{n} = 1$) and so we assume $s > 0$. In this case, we shall now inductively construct primes $\{\mathfrak{q}_a\}_{a \in [s]} \subseteq \mathcal{Q}_j$ with the property that, for each $b \in [s]$, the ideal $\mathfrak{n}_b \coloneqq \prod_{a \in [b]} \mathfrak{q}_a$ is such that $\lambda^*_{\overline{F}}(\mathfrak{n}_b) = \lambda^*_{\overline{F}}(1) - b$. We thus assume that suitable primes $\{\mathfrak{q}_a\}_{a \in [b]}$ have been constructed for some b with $0 \leq b < s$, and we set $\mathfrak{n}_b \coloneqq \prod_{a \in [b]} \mathfrak{q}_a$.

Then, since Hypothesis 4.14 (vi) implies $\chi(\overline{F}, j(i)) \geq 0$, where the integer $j(i)$ is defined in Definition 4.13, one has

$$0 < \lambda^*_{\overline{F}}(1) - b = \lambda^*_{\overline{F}}(\mathfrak{n}_b) = \lambda^*_{\overline{F}}(\mathfrak{n}_b, j(i)) \leq \lambda_{\overline{F}}(\mathfrak{n}_b, j(i)).$$

It follows that the groups $H^1_{\overline{F}(\mathfrak{n}_b)}(k, \overline{A})/\text{Ш}_{\overline{F}, j(i)}(\overline{A})$ and $H^1_{\overline{F}^*(\mathfrak{n}_b)}(k, \overline{B})$ are both nonzero. We may therefore apply Proposition 6.5 (i) in order to fix a prime $\mathfrak{q}_{b+1} \in \mathcal{Q}_j$ such that the localisation maps $H^1_{\overline{F}(\mathfrak{n}_b)}(k, \overline{A}) \to H^1(k_{\mathfrak{q}_{b+1}}, \overline{A})$ and $H^1_{\overline{F}^*(\mathfrak{n}_b)}(k, \overline{B}) \to H^1(k_{\mathfrak{q}_{b+1}}, \overline{B})$ are both nonzero. Given this, the result of [26, Prop. 5.7] implies

$$\lambda^*_{\overline{F}}(\mathfrak{n}_{b+1}) = \lambda^*_{\overline{F}}(\mathfrak{n}_b) - 1 = \lambda^*_{\overline{F}}(1) - (b+1),$$

as required to complete the induction step. This proves the claimed result. □

We show next that, for suitable moduli $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{n}$, the ideals J_i in Definition 4.13 can be used to bound the complexity of the groups $H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{n})}(k, \mathcal{B}) = H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{B})$.

(6.13) Lemma. *Fix pairwise coprime moduli $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{n}$ in $\mathcal{N}_{j(i)}$ for which $H^1_{(\overline{F}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \overline{B})$ vanishes. Then one has $\mathrm{Fitt}^0_{\Lambda}(J_i) \subseteq \Lambda \cdot \varrho_i\big(\mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{B})^*)\big)$.*

Proof. We compare the exact sequence obtained by applying the functor $(-) \otimes_{\mathbb{A}} \Lambda$ to the short exact sequence in Proposition 5.9 (iii) with the short exact sequence obtained from Proposition 5.9 (iii) after replacing $\mathcal{A}$ and $\tilde{\mathcal{F}}$ by A and $\tilde{F}$. In this way, we obtain an exact commutative diagram

$$\begin{array}{ccccc} \mathrm{Tor}_1^{\mathbb{A}}(X(\tilde{\mathscr{F}}), \Lambda) & \longrightarrow & H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{B})^* \otimes_{\mathbb{A}} \Lambda & \longrightarrow & H^1(C(\tilde{\mathscr{F}}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{n}))) \otimes_{\mathbb{A}} \Lambda \\ & & \downarrow & & \downarrow\simeq \\ 0 & \longrightarrow & H^1_{(\tilde{F}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, B)^* & \longrightarrow & H^1(C((\tilde{\mathscr{F}} \otimes_{\mathbb{A}} \Lambda)^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{n}))), \end{array} \tag{6.14}$$

where the vertical isomorphism is induced, via Lemma 2.31 (ii), by the isomorphism in Proposition 5.9 (ii). In addition, the assumed vanishing of $H^1_{(\overline{F}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \overline{B})$ combines with Lemmas 2.9 and 3.37 to imply $H^1_{(\tilde{F}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, B)$ also vanishes. The above diagram therefore gives rise to an exact sequence

$$\mathrm{Tor}_1^{\mathbb{A}}(X(\tilde{\mathscr{F}}), \Lambda) \to H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{B})^* \otimes_{\mathbb{A}} \Lambda \to 0.$$

Upon comparing this sequence to that obtained by applying the functor $(-) \otimes_{R_{j(i)}} \Lambda$ to the analogous sequence with $\mathbb{A}, \tilde{\mathscr{F}}, \Lambda$ and $\mathcal{B}$ replaced by $\mathcal{R}_{j(i)}, \mathscr{F}_{j(i)}, R_{j(i)}$ and $\mathcal{T}^*_{j(i)}(1)$, one obtains an exact commutative diagram

$$\begin{array}{ccccc} \mathrm{Tor}_1^{\mathcal{R}_{j(i)}}(X(\mathscr{F}_{j(i)}), \mathbb{A}) \otimes_{R_{j(i)}} \Lambda & \longrightarrow & H^1_{((\mathcal{F}_{j(i)})^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{T}^*_{j(i)}(1))^* \otimes_{\mathcal{R}_{j(i)}} \Lambda & \longrightarrow & 0 \\ \downarrow\alpha & & \downarrow\beta & & \\ \mathrm{Tor}_1^{\mathbb{A}}(X(\tilde{\mathscr{F}}), \Lambda) & \longrightarrow & H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{B})^* \otimes_{\mathbb{A}} \Lambda & \longrightarrow & 0. \end{array}$$

Here α is the natural map and, similarly to (6.14), the map β is induced by the map

$$\beta' \colon H^1_{((\mathcal{F}_{j(i)})^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{T}^*_{j(i)}(1))^* \to H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{B})^*$$

arising as the restriction (via the exact sequence in Proposition 5.9 (iii)) of the surjective map $H^1(C((\mathscr{F}_{j(i)})^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{n}))) \to H^1(C(\tilde{\mathscr{F}}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{n})))$ induced by Proposition 5.9 (ii). In particular, β is surjective since β' is dual to the injective map

$$H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{B}) \cong H^1_{((\mathcal{F}_{j(i)})^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{T}^*_{j(i)}(1))[\mathfrak{a}_i] \subseteq H^1_{((\mathcal{F}_{j(i)})^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k, \mathcal{T}^*_{j(i)}(1))$$

induced by Lemma 3.37. Now, by definition of $j(i)$, the image of α in the above diagram is equal to the submodule J_i and so the commutativity of the diagram combines with the surjectivity of β to imply the existence of a surjective map $J_i \twoheadrightarrow H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k,\mathcal{B})^* \otimes_{\mathbb{\Lambda}} \Lambda$ of Λ-modules. This surjective map then combines with Lemma 2.11 (ii) and (iv) to imply an inclusion

$$\mathrm{Fitt}^0_{\Lambda}(J_i) \subseteq \mathrm{Fitt}^0_{\Lambda}(H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k,\mathcal{B})^* \otimes_{\mathbb{\Lambda}} \Lambda) = \Lambda \cdot \varrho_i\big(\mathrm{Fitt}^0_{\mathbb{\Lambda}}(H^1_{(\tilde{\mathcal{F}}^{\mathfrak{a}}_{\mathfrak{b}})^*(\mathfrak{n})}(k,\mathcal{B})^*)\big).$$

This proves the claimed result. □

(6.15) Remark. Taking $\mathfrak{a} = \mathfrak{b} = 1$ in Lemma 6.13, one obtains the following useful fact: if $\mathfrak{n}$ is a relative core vertex for $\tilde{\mathcal{F}}$ that belongs to $\mathcal{N}_{j(i)}$, then $\mathrm{Fitt}^0_{\Lambda}(J_i) \subseteq \Lambda \cdot \varrho_i\big(\mathrm{Fitt}^0_{\mathbb{\Lambda}}(H^1_{\tilde{\mathcal{F}}^*(\mathfrak{n})}(k,\mathcal{B})^*)\big)$.

6.2. Controlling Kolyvagin systems via relative core vertices

The aim of this section is to show that if $\mathfrak{n}$ is a relative core vertex for $\tilde{\mathcal{F}}$ that belongs to $\mathcal{N}_{j(i)}$, and κ is any Kolyvagin system in $\mathrm{KS}^t(\tilde{\mathscr{F}})$ that vanishes at $\mathfrak{n}$, then the value of κ at the modulus 1 can be explicitly controlled. This result is stated precisely as Theorem 6.38 and its proof adapts the graph-theoretical analysis of core vertices that forms a key part of the approach of Mazur and Rubin in [74, § 4].

6.2.1. Graphs and paths

We use the following variants of the notion of the graph $\mathcal{X}^0$ of core vertices for Mazur–Rubin structures that are defined in [74, Def. 4.3.6] (see also [26, Def. 5.14]).

(6.16) Definition. *For each $j \in \mathbb{N}$ with $j \geq i$ we define a graph $\mathcal{X}^0_j = \mathcal{X}^0_j(i)$ as follows.*

(i) The vertices of $\mathcal{X}^0_j$ are the relative core vertices for $\tilde{\mathcal{F}}$ that are contained in $\mathcal{N}_j$.

(ii) Vertices $\mathfrak{n}$ and $\mathfrak{n}\mathfrak{q}$ as in (i) are joined by an edge in $\mathcal{X}^0_j$ if and only if the localisation map $H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A}) \to H^1_f(k_{\mathfrak{q}},\overline{A})$ is non-zero.

(6.17) Definition. *A 'path' on $\mathcal{X}^0_j$ is a finite ordered set*

$$U = \{(\mathfrak{n}_1,\mathfrak{q}_1),\dots,(\mathfrak{n}_s,\mathfrak{q}_s)\} \subset \mathcal{N}_j \times \mathcal{Q}_j$$

with the property that, for every $a \in [s-1]$ one has either $\mathfrak{n}_{a+1} = \mathfrak{n}_a\mathfrak{q}_a$ or $\mathfrak{n}_{a+1} = \mathfrak{n}_a/\mathfrak{q}_a$. We set $|U| := s$ and refer to this as the 'length' of U. We also say that moduli $\mathfrak{n}$ and $\mathfrak{n}'$ in $\mathcal{N}_j$ are connected by U if one has $\mathfrak{n} = \mathfrak{n}_1$ and $\mathfrak{n}' = \mathfrak{n}_s$.

We shall need an upper bound on the minimum possible length of paths between certain pairs of vertices on $\mathcal{X}^0_{j(i)}$. To prove such a result we carefully analyse the arguments of [26, Cor. 5.16] in order to determine the length of the paths that are constructed in the latter result. In fact, though our current hypotheses are weaker than those of loc. cit., this analysis doesn't require any essentially new ideas. Nevertheless, since it forms a key part of our argument, for the convenience of the reader we provide a detailed argument in the remainder of this subsection. We therefore start by recalling a result of Sakamoto, Sano and the second author.

(6.18) Lemma ([26, Lem. 5.13]). *Assume $j \geq i$ and fix $\mathfrak{n} \in \mathcal{N}_j$ and $\mathfrak{q} \in \mathcal{Q}_j \setminus V(\mathfrak{n})$.*

(i) If $\mathfrak{n}$ is a relative core vertex for $\tilde{\mathcal{F}}$ and the map $H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A}) \to H^1_f(k_{\mathfrak{q}},\overline{A})$ is non-zero, then $\mathfrak{n}\mathfrak{q}$ is a relative core vertex for $\tilde{\mathcal{F}}$ and $\mathfrak{n}$ and $\mathfrak{n}\mathfrak{q}$ are joined by an edge in $\mathcal{X}^0_j$.

(ii) If $\mathfrak{n}\mathfrak{q}$ is a relative core vertex for $\tilde{\mathcal{F}}$ and the map $H^1_{\overline{F}(\mathfrak{n}\mathfrak{q})}(k,\overline{A}) \to H^1_{\mathrm{tr}}(k_{\mathfrak{q}},\overline{A})$ is non-zero, then $\mathfrak{n}$ is a relative core vertex for $\tilde{\mathcal{F}}$ and $\mathfrak{n}$ and $\mathfrak{n}\mathfrak{q}$ are joined by an edge in $\mathcal{X}^0_j$.

Our next two results are then minor refinements of [26, Lem. 5.14] and [26, Lem. 5.15] respectively in that they also specify an explicit bound on the length of constructed paths.

(6.19) Lemma. *Assume Hypotheses 4.14 and fix $\mathfrak{n} \in \mathcal{N}_{j(i)}$ and $\mathfrak{q} \in \mathcal{Q}_{j(i)}$. Then, if $\mathfrak{n}$ and $\mathfrak{n}\mathfrak{q}$ are both relative core vertices for $\tilde{\mathcal{F}}$, there exists a path from $\mathfrak{n}$ to $\mathfrak{n}\mathfrak{q}$ in $\mathcal{X}^0_{j(i)}$ that is of length at most three.*

Proof. If the localisation map $H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A}) \to H^1_f(k_{\mathfrak{q}},\overline{A})$ is nonzero, then Lemma 6.18 (i) implies that $\mathfrak{n}$ and $\mathfrak{nq}$ are joined by a path of length 1. We may therefore assume this localisation map is zero, and hence that $H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A}) = H^1_{\overline{F}_{\mathfrak{q}}(\mathfrak{n})}(k,\overline{A}) \subseteq H^1_{\overline{F}(\mathfrak{nq})}(k,\overline{A})$. The latter inclusion must therefore be an equality since the fact $\mathfrak{n}$ and $\mathfrak{nq}$ are relative core vertices combines with (6.11) and the first assertion of Lemma 6.3 to imply that

$$\dim_{\Bbbk}(H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A})) = \chi(\overline{F},j(i)) + \dim_{\Bbbk}(Ш_{\overline{F},j(i)}(\overline{A})) = \dim_{\Bbbk}(H^1_{\overline{F}(\mathfrak{nq})}(k,\overline{A})).$$

Since $\chi(\overline{F},j(i)) > 0$ (by Hypotheses 4.14 (vi)), the first equality here implies that the inclusion $Ш_{\overline{F},j(i)}(\overline{A}) \subseteq H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A})$ is strict. In addition, since $H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A}) = H^1_{\overline{F}(\mathfrak{nq})}(k,\overline{A})$, the exact sequence (5.16) (with the data $\Phi = \tilde{\mathcal{F}}, \mathcal{A}, \mathbb{A}, \mathfrak{m}, \mathfrak{n}$ taken to be $\overline{F}, \overline{A}, \Bbbk, \mathfrak{nq}$ and $\mathfrak{q}$) implies that $H^1_{(\overline{F}^*)^{\mathfrak{q}}(\mathfrak{n})}(k,\overline{B})$ does not vanish. We can therefore apply Proposition 6.5 (i) to deduce the existence of a prime $\mathfrak{r}$ in $\mathcal{Q}_{j(i)} \setminus V(\mathfrak{nq})$ for which the localisation maps

$$H^1_{\overline{F}(\mathfrak{n})}(k,\overline{A}) \to H^1(k_{\mathfrak{r}},\overline{A}) \quad \text{and} \quad H^1_{(\overline{F}^*)^{\mathfrak{q}}(\mathfrak{n})}(k,\overline{B}) \to H^1(k_{\mathfrak{r}},\overline{B})$$

are both nonzero. By using the argument of [26, Lem. 5.14], one then concludes the existence of a path in $\mathcal{X}^0_{j(i)}$ of the form $\mathfrak{n} \to \mathfrak{nr} \to \mathfrak{nrq} \leftarrow \mathfrak{nq}$.
At this stage, we have proved that, in all cases, $\mathfrak{n}$ and $\mathfrak{nq}$ are connected by a path of length at most three, as required. □

(6.20) Lemma. *Assume Hypotheses 4.14. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be relative core vertices for $\tilde{\mathcal{F}}$ in $\mathcal{N}_{j(i)}$, and fix primes $\mathfrak{q}_1 \in V(\mathfrak{n}_1)$ and $\mathfrak{q}_2 \in V(\mathfrak{n}_2)$ such that neither $\mathfrak{n}_1/\mathfrak{q}_1$ and $\mathfrak{n}_2/\mathfrak{q}_2$ are relative core vertices for $\tilde{\mathcal{F}}$. Then there exists a prime $\mathfrak{r}$ in $\mathcal{Q}_{j(i)} \setminus V(\mathfrak{n}_1\mathfrak{n}_2)$ with the following properties.*

(i) Both $\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1$ and $\mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2$ are relative core vertices for $\tilde{\mathcal{F}}$.

(ii) For $l \in \{1,2\}$, there exists a path in $\mathcal{X}^0_{j(i)}$ between $\mathfrak{n}_l$ and $\mathfrak{n}_l\mathfrak{r}/\mathfrak{q}_l$ of length at most four.

Proof. Proposition 6.5 (ii) combines with Hypothesis 4.14 (iv) to imply the existence of a prime $\mathfrak{r}$ in $\mathcal{Q}_{j(i)} \setminus V(\mathfrak{n}_1\mathfrak{n}_2)$ such that the maps

$$H^1_{\overline{F}(\mathfrak{n}_l)}(k,\overline{A}) \to H^1(k_{\mathfrak{r}},\overline{A}) \quad \text{and} \quad H^1_{\overline{F}(\mathfrak{n}_l/\mathfrak{q}_l)}(k,\overline{A}) \to H^1(k_{\mathfrak{r}},\overline{A})$$

are non-zero for both $l \in \{1,2\}$. Lemma 6.18 (i) then implies that $\mathfrak{n}_l$ and $\mathfrak{n}_l\mathfrak{r}$ are directly connected by an edge in $\mathcal{X}^0_{j(i)}$. The proof of [26, Lem. 5.15] moreover shows that $\mathfrak{n}_l\mathfrak{r}/\mathfrak{q}_l$ is a relative core vertex for $\tilde{\mathcal{F}}$, and so Lemma 6.19 implies the existence of a path in $\mathcal{X}^0_{j(i)}$ between $\mathfrak{n}_l\mathfrak{r}$ and $\mathfrak{n}_l\mathfrak{r}/\mathfrak{q}_l$ of length at most three. In total, therefore, there exists a path between $\mathfrak{n}_l$ and $\mathfrak{n}_l\mathfrak{r}/\mathfrak{q}_l$ in $\mathcal{X}^0_{j(i)}$ of length at most four. □

We can now prove our main observation concerning path lengths.

(6.21) Proposition. *Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be relative core vertices for $\tilde{\mathcal{F}}$ in $\mathcal{N}_{j(i)}$ for which $\nu(\mathfrak{n}_1) = \nu(\mathfrak{n}_2) = \lambda^*_{\overline{F}}(1)$. Then, if Hypotheses 4.14 is valid, there exists a path in $\mathcal{X}^0_{j(i)}$ between $\mathfrak{n}_1$ and $\mathfrak{n}_2$ of length at most $8 \cdot (\lambda^*_{\overline{F}}(1) - \nu(\gcd(\mathfrak{n}_1,\mathfrak{n}_2)))$.*

Proof. We argue by induction on the non-negative integer

$$\mu_{\overline{F}}(\mathfrak{n}_1,\mathfrak{n}_2) := \lambda^*_{\overline{F}}(1) - \nu(\gcd(\mathfrak{n}_1,\mathfrak{n}_2)).$$

Firstly, if $\mu_{\overline{F}}(\mathfrak{n}_1,\mathfrak{n}_2) = 0$, and hence $\nu(\gcd(\mathfrak{n}_1,\mathfrak{n}_2)) = \lambda^*_{\overline{F}}(1)$, then one must have $\mathfrak{n}_1 = \mathfrak{n}_2$ since, by assumption, $\nu(\mathfrak{n}_1)$ and $\nu(\mathfrak{n}_2)$ are also both equal to $\lambda^*_{\overline{F}}(1)$. This proves the induction base. We therefore assume $\mu_{\overline{F}}(\mathfrak{n}_1,\mathfrak{n}_2) > 0$, and hence that $\mathfrak{n}_1 \neq \mathfrak{n}_2$, and then fix primes $\mathfrak{q}_1 \in V(\mathfrak{n}_1/\gcd(\mathfrak{n}_1,\mathfrak{n}_2))$ and $\mathfrak{q}_2 \in V(\mathfrak{n}_2/\gcd(\mathfrak{n}_1,\mathfrak{n}_2))$. Now, by [26, Cor. 5.11], one knows that any relative core vertex $\mathfrak{m}$ for $\tilde{\mathcal{F}}$ satisfies $\nu(\mathfrak{m}) \geq \lambda^*_{\overline{F}}(1)$. In particular, since $\nu(\mathfrak{n}_l/\mathfrak{q}_l) = \nu(\mathfrak{n}_l) - 1 = \lambda^*_{\overline{F}}(1) - 1$ for $l \in \{1,2\}$, neither $\mathfrak{n}_1/\mathfrak{q}_1$ nor $\mathfrak{n}_2/\mathfrak{q}_2$ can be a relative core vertex for $\tilde{\mathcal{F}}$. Lemma

6.20 therefore implies the existence of a prime $\mathfrak{r}$ in $\mathcal{Q}_{j(i)} \setminus V(\mathfrak{n}_1\mathfrak{n}_2)$ such that, for both $l \in \{1,2\}$, there exists a path in $\mathcal{X}^0_{j(i)}$ between $\mathfrak{n}_l$ and $\mathfrak{n}_l\mathfrak{r}/\mathfrak{q}_l$ of length at most four. In addition, one has

$$\begin{aligned}\mu_{\overline{F}}(\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1, \mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2) &= \lambda^*_{\overline{F}}(1) - \nu(\gcd(\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1, \mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2)) \\ &= \lambda^*_{\overline{F}}(1) - \nu(\gcd(\mathfrak{n}_1, \mathfrak{n}_2)) - 1 \\ &= \mu_{\overline{F}}(\mathfrak{n}_1, \mathfrak{n}_2) - 1,\end{aligned}$$

and so, by the induction hypothesis, the vertices $\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1$ and $\mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2$ are connected in $\mathcal{X}^0_{j(i)}$ by a path of length at most $8 \cdot \mu_{\overline{F}}(\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1, \mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2)$. By concatenating these three paths, we have therefore obtained a path in $\mathcal{X}^0_{j(i)}$ from $\mathfrak{n}_1$ to $\mathfrak{n}_2$ (via $\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1$ and $\mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2$) of length at most

$$4 + 8 \cdot \mu_{\overline{F}}(\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1, \mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2) + 4 = 8(\mu_{\overline{F}}(\mathfrak{n}_1\mathfrak{r}/\mathfrak{q}_1, \mathfrak{n}_2\mathfrak{r}/\mathfrak{q}_2) + 1) = 8 \cdot \mu_{\overline{F}}(\mathfrak{n}_1, \mathfrak{n}_2),$$

as required. □

6.2.2. Moving along the graph

Given two values $\kappa_\mathfrak{n}$ and $\kappa_{\mathfrak{n}\mathfrak{q}}$ of a system κ in $\mathrm{KS}^t(\tilde{\mathscr{F}})$, the defining relation of Kolyvagin systems relate their images under the maps that are respectively induced on biduals by $\psi^{\mathrm{fs}}_\mathfrak{q}$ and $v_\mathfrak{q}$. In order to be able to relate the values $\kappa_\mathfrak{n}$ and $\kappa_{\mathfrak{n}\mathfrak{q}}$ themselves, it is therefore crucial to control the kernels of these maps. This is achieved by the following result that uses the integer $\chi_\mathfrak{F}$ fixed at the beginning of § 4.3.

(6.22) Lemma. *Fix $t \in \mathbb{N}_0$ such that $t_\mathfrak{F} := t + \chi_\mathfrak{F} > 0$. Then, for each $\mathfrak{n} \in \mathcal{N}_{j(i)}$ and $\mathfrak{q} \in \mathcal{P} \setminus V(\mathfrak{n})$, the kernels of both maps*

$$\check{\psi}^{\mathrm{fs}}_\mathfrak{q} \colon {\bigcap}^{t_\mathfrak{F}}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A}) \to {\bigcap}^{t_\mathfrak{F}-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})$$

$$\check{v}_\mathfrak{q} \colon {\bigcap}^{t_\mathfrak{F}}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}(\mathfrak{n}\mathfrak{q})}(k, \mathcal{A}) \to {\bigcap}^{t_\mathfrak{F}-1}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})$$

are annihilated by $\mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{q}}(\mathfrak{n})}(k, \mathcal{B})^*) \cdot \mathrm{Fitt}^t_{\mathbb{A}}(X(\tilde{\mathscr{F}}))$.

Proof. It follows from the exact sequences (5.16) and (5.17) (with $\Phi = \tilde{\mathscr{F}}$ and $\mathfrak{n} = \mathfrak{q}, \mathfrak{m} = \mathfrak{n}\mathfrak{q}$, respectively $\mathfrak{a} = \mathfrak{q}$ and $\mathfrak{m} = \mathfrak{n}$) that $H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})$ coincides with the kernel of both of the maps $\check{v}_\mathfrak{q} \colon H^1_{\tilde{\mathscr{F}}(\mathfrak{n}\mathfrak{q})}(k, \mathcal{A}) \to \mathbb{A}$ and $\check{\psi}^{\mathrm{fs}}_\mathfrak{q} \colon H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A}) \to \mathbb{A}$. Lemma 2.17 (i) therefore implies that ${\bigcap}^{t_\mathfrak{F}}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})$ is the kernel of both of the displayed maps, and so we must show the stated product ideal annihilates the latter module. In addition, by Lemma 2.11 (v), the module ${\bigwedge}^{t_\mathfrak{F}}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})^*$, and hence also its $\mathbb{A}$-linear dual ${\bigcap}^{t_\mathfrak{F}}_{\mathbb{A}} H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})$, is annihilated by $\mathrm{Fitt}^{t_\mathfrak{F}-1}_{\mathbb{A}}(H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})^*)$ and so we are reduced to showing that

$$\mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{q}}(\mathfrak{n})}(k, \mathcal{B})^*) \cdot \mathrm{Fitt}^t_{\mathbb{A}}(X(\tilde{\mathscr{F}})) \subseteq \mathrm{Fitt}^{t_\mathfrak{F}-1}_{\mathbb{A}}(H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})^*). \tag{6.23}$$

To do this, we use the complex $C(\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n}))$ from Proposition 5.9. In particular, from Proposition 5.9 (i) and (iii) one has $\chi_{\mathbb{A}}(C(\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n}))) = \chi_\mathfrak{F} - 1$ and $H^0(C(\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n}))) = H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})$. By applying Lemma 2.36 with $C = C(\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n}))$ and $Y = (0)$, one therefore has

$$\mathrm{Fitt}^{t_\mathfrak{F}-1}_{\mathbb{A}}(H^1_{\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})}(k, \mathcal{A})^*) = \mathrm{Fitt}^t_{\mathbb{A}}(H^1(C(\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})))).$$

To deduce the required equality (6.23), we then need only note that Lemma 2.11 (ii) applies to the exact sequence in Proposition 5.9 (iii) (with $\mathfrak{a}$ and $\mathfrak{b}$ taken to be $\mathfrak{q}$ and 1) to imply

$$\mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{q}}(\mathfrak{n})}(k, \mathcal{B})^*) \cdot \mathrm{Fitt}^t_{\mathbb{A}}(X(\tilde{\mathscr{F}})) \subseteq \mathrm{Fitt}^t_{\mathbb{A}}(H^1(C(\tilde{\mathscr{F}}_\mathfrak{q}(\mathfrak{n})))). \qquad \square$$

Lemma 6.22 has the following consequence concerning the values of Kolyvagin systems at moduli that are connected by paths on the graph $\mathcal{X}^0_{j(i)}$.

(6.24) Lemma. *Fix $t \in \mathbb{N}_0$ such that $t_{\mathfrak{F}} := t + \chi_{\mathfrak{F}} > 0$. Let $\kappa \in \mathrm{KS}^{t_{\mathfrak{F}}}(\tilde{\mathscr{F}})$ be a Kolyvagin system of rank $t_{\mathfrak{F}}$ for $\tilde{\mathscr{F}}$ and $\mathfrak{n}$ a relative core vertex for $\tilde{\mathcal{F}}$ in $\mathcal{N}_{j(i)}$ for which $\kappa_{\mathfrak{n}} = 0$. Then, if U is a path that connects $\mathfrak{n}$ to $\mathfrak{n}'$ in $\mathcal{X}^0_{j(i)}$, one has*

$$\big(\prod\nolimits_{(\mathfrak{m},\mathfrak{q})\in U} \mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{q}}(\mathfrak{m})}(k,\mathcal{B})^*)\big) \cdot \mathrm{Fitt}^t_{\mathbb{A}}(X(\tilde{\mathscr{F}}))^{|U|} \cdot \kappa_{\mathfrak{n}'} = (0).$$

Proof. We will prove the claim by induction on the length $s = |U|$ of U. Let us therefore assume that the claim has already been proved for all paths of length at most $s-1$.
Writing $U = ((\mathfrak{n}_1,\mathfrak{q}_1),\dots,(\mathfrak{n}_s,\mathfrak{q}_s))$, we then see that $\mathfrak{n}_1$ and $\mathfrak{n}_{s-1}$ are connected by a path of length $s-1$. By the induction hypothesis it therefore follows that

$$\big(\prod\nolimits_{l\in[s-1]} \mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{q}_l}(\mathfrak{n}_l)}(k,\mathcal{B})^*)\big) \cdot \mathrm{Fitt}^t_{\mathbb{A}}(X(\tilde{\mathscr{F}}))^{s-1} \cdot \kappa_{\mathfrak{n}_{s-1}} = (0). \tag{6.25}$$

Now, as $\mathfrak{n}' = \mathfrak{n}_s$ and $\mathfrak{n}_{s-1}$ are connected by a path of length one, we have either $\mathfrak{n}' = \mathfrak{n}_{s-1}\mathfrak{q}_{s-1}$ or $\mathfrak{n}' = \mathfrak{n}_{s-1}/\mathfrak{q}_{s-1}$ and we consider these cases separately.
We first assume $\mathfrak{n}' = \mathfrak{n}_{s-1}\mathfrak{q}_{s-1}$. In this case, the defining relation of Kolyvagin systems implies

$$v_{\mathfrak{q}_{s-1}}(\kappa_{\mathfrak{n}'}) = \psi^{\mathrm{fs}}_{\mathfrak{q}_{s-1}}(\kappa_{\mathfrak{n}'/\mathfrak{q}_{s-1}}) = \psi^{\mathrm{fs}}_{\mathfrak{q}_{s-1}}(\kappa_{\mathfrak{n}_{s-1}}).$$

This equality then combines with (6.25) to imply an inclusion

$$\big(\prod\nolimits_{l\in[s-1]} \mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{q}_l}(\mathfrak{n}_l)}(k,\mathcal{B})^*)\big) \cdot \mathrm{Fitt}^t_{\mathbb{A}}(X(\tilde{\mathscr{F}}))^{s-1} \cdot \kappa_{\mathfrak{n}'} \subseteq \ker(v_{\mathfrak{q}_{s-1}}).$$

In particular, since (the second assertion of) Lemma 6.22 implies that the kernel of $v_{\mathfrak{q}_{s-1}}$ is annihilated by the product of $\mathrm{Fitt}^0_{\mathbb{A}}(H^1_{(\tilde{\mathcal{F}}^*)^{\mathfrak{q}_{s-1}}(\mathfrak{n}_{s-1})}(k,\mathcal{B})^*)$ and $\mathrm{Fitt}^t_{\mathbb{A}}(X(\tilde{\mathscr{F}}))$, the claimed equality is clear in this case.
We now assume $\mathfrak{n}' = \mathfrak{n}_{s-1}/\mathfrak{q}_{s-1}$. In this case, the relevant Kolyvagin system relation asserts

$$\psi^{\mathrm{fs}}_{\mathfrak{q}_{s-1}}(\kappa_{\mathfrak{n}'}) = v_{\mathfrak{q}_{s-1}}(\kappa_{\mathfrak{n}'\mathfrak{q}_{s-1}}) = v_{\mathfrak{q}_{s-1}}(\kappa_{\mathfrak{n}_{s-1}}).$$

Then, just as above, this equality can be combined with (6.25) and (the first assertion of) Lemma 6.22 to deduce the validity of the claimed equality. This therefore concludes the inductive step, thereby proving the claimed result. □

6.2.3. Bounding dimensions

In this section, we bound the dimensions of $\mathbb{K}$-spaces that arise in subsequent arguments.

(6.26) Lemma. *Let $\mathfrak{n}$ and Q be coprime moduli in $\mathcal{N}_{j(i)}$. Then, if the homomorphism*

$$\big(H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})/\text{Ш}_{\tilde{\mathscr{F}},j(i)}(\mathcal{A})\big)[\mathcal{M}_i] \xrightarrow{(\breve{\psi}^{\mathrm{fs}}_{\mathfrak{q}})_{\mathfrak{q}}} \bigoplus\nolimits_{\mathfrak{q}\in V(Q)} \mathbb{A}$$

is injective, one has $H^1_{\tilde{\mathscr{F}}_Q(\mathfrak{n})}(k,\mathcal{A}) = \text{Ш}_{\tilde{\mathscr{F}},j(i)}(\mathcal{A})$.

Proof. The relevant case of the long exact sequence (5.17) implies that the kernel of the displayed map is equal to $\big(H^1_{\tilde{\mathscr{F}}_Q(\mathfrak{n})}(k,\mathcal{A})/\text{Ш}_{\tilde{\mathscr{F}},j(i)}(\mathcal{A})\big)[\mathcal{M}_i]$. The given assumption therefore implies that this module vanishes and hence, by Lemma 2.9, that $H^1_{\tilde{\mathscr{F}}_Q(\mathfrak{n})}(k,\mathcal{A})/\text{Ш}_{\tilde{\mathscr{F}},j(i)}(\mathcal{A})$ itself vanishes, as claimed. □

For each modulus $\mathfrak{n}$ in $\mathcal{N}_{j(i)}$, we set

$$\begin{aligned}\alpha_i(\mathfrak{n}) &:= \dim_{\mathbb{K}}\big(\big(H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k,\mathcal{A})/\text{Ш}_{\tilde{\mathscr{F}},j(i)}(\mathcal{A})\big)[\mathcal{M}_i]\big) \\ &= \dim_{\mathbb{K}}\big(\big(H^1_{\tilde{\mathcal{F}}(\mathfrak{n})}(k,\mathcal{A})/\text{Ш}_{\tilde{\mathcal{F}},j(i)}(\mathcal{A})\big)[\mathcal{M}_i]\big),\end{aligned} \tag{6.27}$$

where the equality is a consequence of the isomorphism in Lemma 6.1 (iii). This quantity constitutes an upper bound for the minimal possible value of $\nu(Q)$ as Q ranges over moduli in $\mathcal{N}_{j(i)}$ that are coprime to $\mathfrak{n}$ and such that the displayed map in Lemma 6.26 is injective. The following result establishes some crucial properties of these bounds.

(6.28) Theorem. *The following claims are valid.*

(i) For $\mathfrak{n} \in \mathcal{N}_{j(i)}$ *and* $\mathfrak{q} \in \mathcal{Q}_{j(i)} \setminus V(\mathfrak{n})$*, one has* $|\alpha_i(\mathfrak{n}\mathfrak{q}) - \alpha_i(\mathfrak{n})| \le 1$*.*

(ii) There exists an increasing function $\Phi_{\mathscr{F}_k} : \mathbb{N}_0 \to \mathbb{N}_0$ *such that, for every* $(i, \mathfrak{n}) \in \mathbb{N} \times \mathcal{N}_{j(i)}$*, one has* $\alpha_i(\mathfrak{n}) \le \Phi_{\mathscr{F}_k}(\nu(\mathfrak{n}))$*.*

Proof. To prove (i), we fix an $\mathbb{\Lambda}$-submodule W of $H^1_{\tilde{\mathscr{F}}_{\mathfrak{q}}(\mathfrak{n})}(k, \mathcal{A})$. Then, for the modulus $\mathfrak{n}' = \mathfrak{n}$, respectively $\mathfrak{n}' = \mathfrak{q}\mathfrak{n}$, the exact sequence (5.14) with $(\Phi, \mathfrak{n}, \mathfrak{m})$ taken to be $(\tilde{\mathscr{F}}(\mathfrak{n}), 1, \mathfrak{q})$, respectively (5.15) with $(\Phi, \mathfrak{m})$ taken to be $(\tilde{\mathscr{F}}(\mathfrak{n}), \mathfrak{q})$, gives an exact sequence of $\mathbb{\Lambda}$-modules

$$0 \to H^1_{\tilde{\mathscr{F}}(\mathfrak{n}')}(k, \mathcal{A})/W \to H^1_{\tilde{\mathscr{F}}^{\mathfrak{q}}(\mathfrak{n})}(k, \mathcal{A})/W \to \mathbb{\Lambda}.$$

Applying the functor $(-)[\mathcal{M}_i]$ to this sequence, one obtains an exact sequence of $\mathbb{K}$-modules

$$0 \to \big(H^1_{\tilde{\mathscr{F}}(\mathfrak{n}')}(k, \mathcal{A})/W\big)[\mathcal{M}_i] \to \big(H^1_{\tilde{\mathscr{F}}^{\mathfrak{q}}(\mathfrak{n})}(k, \mathcal{A})/W\big)[\mathcal{M}_i] \to \mathbb{K},$$

from which one deduces that

$$\big|\dim_{\mathbb{K}}\big(\big(H^1_{\tilde{\mathscr{F}}(\mathfrak{n}\mathfrak{q})}(k, \mathcal{A})/W\big)[\mathcal{M}_i]\big) - \dim_{\mathbb{K}}\big(\big(H^1_{\tilde{\mathscr{F}}(\mathfrak{n})}(k, \mathcal{A})/W\big)[\mathcal{M}_i]\big)\big| \le 1. \tag{6.29}$$

Claim (i) now follows upon setting $W = Ш_{\tilde{\mathscr{F}}, j(i)}(\mathcal{A})$.

To prove (ii), we lighten notation by setting $Ш_s := Ш_{\mathscr{F}_s, j(s)}(\mathcal{T}_s)$ for each $s \ge 0$. Then, upon applying the functor $\mathrm{Hom}_{\mathcal{R}_s}(\mathbb{K}, -)$ to the tautological short exact sequence

$$0 \to Ш_s \to H^1_{\mathscr{F}_s(\mathfrak{n})}(k, \mathcal{T}_s) \to H^1_{\mathscr{F}_s(\mathfrak{n})}(k, \mathcal{T}_s)/Ш_s \to 0$$

we obtain an exact sequence of $\mathbb{K}$-modules

$$H^1_{\mathscr{F}_s(\mathfrak{n})}(k, \mathcal{T}_s)[\mathcal{M}_s] \to \big(H^1_{\mathscr{F}_s(\mathfrak{n})}(k, \mathcal{R}_s)/Ш_s\big)[\mathcal{M}_s] \to \mathrm{Ext}^1_{\mathcal{R}_s}(\mathbb{K}, Ш_s). \tag{6.30}$$

To prove (ii), it is therefore suffices to provide suitable bounds on the $\mathbb{K}$-dimensions of the outer terms in this exact sequence.

For the first module, we note that Proposition 5.9 (i), (ii) and (iii) allow us to apply Lemma 2.31 (i) (with $C_n = C(\mathscr{F}_n(\mathfrak{n})), a = 0$ and $b = 1$) to deduce the existence of a natural isomorphism of (finite) $\mathbb{K}$-modules $H^1_{\mathscr{F}_s(\mathfrak{n})}(k, \mathcal{T}_s)[\mathcal{M}_s] \cong H^1_{\mathscr{F}_0(\mathfrak{n})}(k, \overline{\mathcal{T}})$. We write $\beta(\mathfrak{n})$ for the $\mathbb{K}$-dimension of the latter module. Then, for each modulus $\mathfrak{m} \in \mathcal{N}_{j(s)}$ and prime $\mathfrak{q} \in \mathcal{Q}_{j(s)} \setminus V(\mathfrak{m})$, we can take $W = (0)$ in (6.29) (with $\tilde{\mathscr{F}} = \mathscr{F}_i$, $\mathcal{A}$ and $\mathcal{M}_i$ replaced by $\mathscr{F}_0, \overline{\mathcal{T}}$ and $\mathcal{M}_0$) in order to deduce an inequality $|\beta(\mathfrak{m}\mathfrak{q}) - \beta(\mathfrak{m})| \le 1$. By combining this with an induction on $\nu(\mathfrak{n})$, one can then prove that

$$\dim_{\mathbb{K}}\big(H^1_{\mathscr{F}_s(\mathfrak{n})}(k, \mathcal{T}_s)[\mathcal{M}_s]\big) = \beta(\mathfrak{n}) \le \nu(\mathfrak{n}) + \beta(1) = \nu(\mathfrak{n}) + \dim_{\mathbb{K}}\big(H^1_{\mathscr{F}_0}(k, \overline{\mathcal{T}})\big). \tag{6.31}$$

To bound the $\mathbb{K}$-dimension of $\mathrm{Ext}^1_{\mathcal{R}_s}(\mathbb{K}, Ш_s)$ we note that the derived Tensor-Hom adjunction isomorphism

$$\mathrm{RHom}_{\mathcal{R}_s}(\mathbb{K}, \mathrm{RHom}_{\mathcal{R}_s}(Ш_s^*, \mathcal{R}_s)) \xrightarrow{\simeq} \mathrm{RHom}_{\mathcal{R}_s}(\mathbb{K} \otimes^{\mathbb{L}}_{\mathcal{R}_s} Ш_s^*, \mathcal{R}_s)$$

in $D(\mathbb{K})$ (cf. [111, Th. 10.8.7]) induces, on cohomology groups in degree one, an isomorphism

$$\mathrm{Ext}^1_{\mathcal{R}_s}(\mathbb{K}, Ш_s) \cong \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, Ш_s^*)^* \tag{6.32}$$

of $\mathbb{K}$-modules. To study this module, we use the exact commutative diagram of $\mathcal{R}_s$-modules

$$\begin{array}{ccccccc}
0 \longrightarrow & Ш_s & \longrightarrow & H^1_{\mathscr{F}_s}(k, \mathcal{T}_s) & \longrightarrow & \bigoplus_{\mathfrak{q} \in \mathcal{Q}_{j(s)}} H^1_f(k_{\mathfrak{q}}, \mathcal{T}_s) \\
& \downarrow & & \downarrow \simeq & & \downarrow \\
0 \longrightarrow & Ш_{s+1}[\mathfrak{a}_s] & \longrightarrow & H^1_{\mathscr{F}_{s+1}}(k, \mathcal{T}_{s+1})[\mathfrak{a}_s] & \longrightarrow & \bigoplus_{\mathfrak{q} \in \mathcal{Q}_{j(s+1)}} H^1_f(k_{\mathfrak{q}}, \mathcal{T}_{s+1})[\mathfrak{a}_s].
\end{array}$$

Here the rows follow directly from the respective definitions of $Ш_s$ and $Ш_{s+1}$ and (in the second case) the left exactness of the functor $(-)[\mathfrak{a}_s]$. The central vertical map is the isomorphism that is induced, via Lemma 2.31 (i) (and the result of Proposition 5.9), by the fixed isomorphism $\mathcal{R}_{s+1}[\mathfrak{a}_s] \cong \mathcal{R}_s$. Finally, the right hand vertical map is the projection induced by the isomorphisms $H^1_f(k_{\mathfrak{q}}, \mathcal{T}_{s+1})[\mathfrak{a}_s] \cong H^1_f(k_{\mathfrak{q}}, \mathcal{T}_s)$ for $\mathfrak{q} \in \mathcal{Q}_{j(s+1)} \subseteq \mathcal{Q}_{j(s)}$. In particular, the second square in the diagram commutes and so the indicated dashed map exists in order to make the whole diagram commutative. From the commutativity of the diagram, it follows that the latter map is injective and hence, upon taking the dual of the first square in the diagram and using Lemma 2.8 (iii), we obtain a commutative diagram of surjective maps of $\mathcal{R}_{s+1}$-modules

$$\begin{array}{ccccc} H^1_{\mathscr{F}_{s+1}}(k,\mathcal{T}_{s+1})^* \otimes_{\mathcal{R}_{s+1}} \mathcal{R}_s & \xrightarrow{\cong} & (H^1_{\mathscr{F}_{s+1}}(k,\mathcal{T}_{s+1})[\mathfrak{a}_s])^* & \xrightarrow{\cong} & H^1_{\mathscr{F}_s}(k,\mathcal{T}_s)^* \\ \downarrow\!\!\!\twoheadrightarrow & & & & \downarrow\!\!\!\twoheadrightarrow \\ Ш^*_{s+1} \otimes_{\mathcal{R}_{s+1}} \mathcal{R}_s & \xrightarrow{\cong} & \big(Ш_{s+1}[\mathfrak{a}_s]\big)^* & \twoheadrightarrow & Ш^*_s \end{array} \tag{6.33}$$

Since all modules here are finite, exactness is preserved upon passing to the inverse limit (over s) and so one obtains a surjective map of $\mathcal{R}$-modules

$$\varprojlim_{s\in\mathbb{N}} H^1_{\mathscr{F}_s}(k,\mathcal{T}_s)^* \twoheadrightarrow M := \varprojlim_s Ш^*_s,$$

in which the limits are defined with respect to the maps induced by the respective rows of (6.33). In particular, since the upper row of (6.33) is an isomorphism, Nakayama's Lemma implies $\varprojlim_{s\in\mathbb{N}} H^1_{\mathscr{F}_s}(k,\mathcal{T}_s)^*$, and hence also M, is a finitely generated $\mathcal{R}$-module. Thus, if we set $M_s := M \otimes_{\mathcal{R}} \mathcal{R}_s$ for $s \in \mathbb{N}$, the natural map $M \to \varprojlim_{s\in\mathbb{N}} M_s$ is an isomorphism. In addition, since the natural projection $\kappa_s\colon M \to Ш^*_s$ is surjective, setting $N_s := \ker(\mathcal{R}_s \otimes_{\mathcal{R}} \kappa_s)$ gives a tautological short exact sequence of $\mathcal{R}_s$-modules

$$0 \to N_s \to M_s \xrightarrow{\mathcal{R}_s\otimes_{\mathcal{R}}\kappa_s} Ш^*_s \to 0.$$

Since each N_s is finite, exactness is preserved when passing to the inverse limit over s and so $\varprojlim_{s\in\mathbb{N}} N_s$ vanishes. Then, as $\mathbb{K}$ is finitely presented as an $\mathcal{R}$-module, one also has

$$\varprojlim_{s\in\mathbb{N}}(\mathbb{K}\otimes_{\mathcal{R}_s} N_s) = \varprojlim_{s\in\mathbb{N}}(\mathbb{K}\otimes_{\mathcal{R}} N_s) \cong \mathbb{K}\otimes_{\mathcal{R}} \varprojlim_{s\in\mathbb{N}} N_s = (0).$$

In particular, by applying the functor $\mathbb{K}\otimes_{\mathcal{R}_s}(-)$ to the above short exact sequence and then taking inverse limits over s, we obtain an exact sequence

$$\varprojlim_{s\in\mathbb{N}} \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, M_s) \longrightarrow \varprojlim_{s\in\mathbb{N}} \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, Ш^*_s) \longrightarrow \varprojlim_{s\in\mathbb{N}}(\mathbb{K}\otimes_{\mathcal{R}_s} N_s) = (0),$$

and hence, upon taking duals, a composite injective map

$$\varinjlim_s \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, Ш^*_s)^* \cong \big(\varprojlim_s \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, Ш^*_s)\big)^\vee \hookrightarrow \big(\varprojlim_s \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, M_s)\big)^\vee \cong \mathrm{Tor}_1^{\mathcal{R}}(\mathbb{K}, M)^\vee,$$

where the second isomorphism follows from Lemma 2.32 (ii) with R taken to be $\mathcal{R}$ and each S_n to be $\mathbb{K}$. In addition, from Lemma 6.34 below, the $\mathbb{K}$-dimension of the kernel of the direct limit $\mathrm{Tor}_1^{\mathcal{R}_i}(\mathbb{K}, Ш^*_i)^* \to \varinjlim_s \mathrm{Tor}_1^{\mathcal{R}_j}(\mathbb{K}, Ш^*_s)^*$ of the homomorphisms induced by Lemma 2.32(i) is at most $\mathrm{rk}(k(\mathcal{T})_\infty/k)\cdot\dim_{\mathbb{K}}(\overline{\mathcal{T}})$, where $\mathrm{rk}(k(\mathcal{T})_\infty/k)$ is the 'rank' of $\mathrm{Gal}(k(\mathcal{T})_\infty/k)$ as specifed below. Taken together, these observations imply that

$$\dim_{\mathbb{K}}\big(\mathrm{Tor}_1^{\mathcal{R}_i}(\mathbb{K}, Ш^*_i)^*\big) \le \mathrm{rk}(k(\mathcal{T})_\infty/k)\cdot\dim_{\mathbb{K}}(\overline{\mathcal{T}}) + \dim_{\mathbb{K}}\big(\mathrm{Tor}_1^{\mathcal{R}}(\mathbb{K}, M)^\vee\big).$$

We now define the function $\Phi_{\mathscr{F}_k}\colon \mathbb{N}_0 \to \mathbb{N}_0$ by setting

$$\Phi_{\mathscr{F}_k}(m) := m + \dim_{\mathbb{K}}\big(H^1_{\mathscr{F}_0}(k,\overline{\mathcal{T}})\big) + \mathrm{rk}(k(\mathcal{T})_\infty/k)\cdot\dim_{\mathbb{K}}(\overline{\mathcal{T}}) + \dim_{\mathbb{K}}\big(\mathrm{Tor}_1^{\mathcal{R}}(\mathbb{K}, M)^\vee\big).$$

This function is increasing and, by combining the last inequality with the isomorphism (6.32), the inequality (6.31) and the exact sequence (6.30), one checks that $\alpha_i(\mathfrak{n}) \le \Phi_{\mathscr{F}_k}(\nu(\mathfrak{n}))$ for every $\mathfrak{n} \in \mathcal{N}_{j(i)}$, as required to prove (ii). □

Before stating the next result we recall that, under Hypothesis 4.14 (vii), $\mathrm{Gal}(k(\mathcal{T})_\infty/k)$ is a compact p-adic analytic group. In particular, [32, Th. 9.38(ii)] implies that the Sylow p-subgroups of $\mathrm{Gal}(k(\mathcal{T})_\infty/k)$ have a common finite rank (as pro-p groups) and we denote this by $\mathrm{rk}(k(\mathcal{T})_\infty/k)$.

(6.34) Lemma. *Set* $Ш_s := Ш_{\mathscr{F}_s, j(s)}(\mathcal{T}_s)$ *for each* $s \in \mathbb{N}$*. Then, for* $i \in \mathbb{N}$*, one has*

$$\dim_{\mathbb{K}}\big(\ker\big(\mathrm{Tor}_1^{\mathcal{R}_i}(\mathbb{K}, Ш_i^*)^* \xrightarrow{\Theta_i} \varinjlim_j \mathrm{Tor}_1^{\mathcal{R}_j}(\mathbb{K}, Ш_j^*)^*\big)\big) \leq \mathrm{rk}(k(\mathcal{T})_\infty/k) \cdot \dim_{\mathbb{K}}(\overline{\mathcal{T}}),$$

where the map Θ_i *is the direct limit of the homomorphisms induced by Lemma 2.32.*

Proof. For integers s and t with $t \geq s \geq i$ we set

$$Ш_{s,t} := Ш_{\mathscr{F}_s, t}(\mathcal{T}_s), \;\; \widetilde{Ш}_{s,t} := Ш_{\mathcal{F}_s, t}(\mathcal{T}_s) \text{ and } \mathfrak{X}_s := \mathfrak{X}_{\mathcal{F}_s, s}(\mathcal{T}_s)$$

(so that $Ш_s = Ш_{s,j(s)}$). Then $Ш_{s,s} \subseteq Ш_{s,t}$ and $\widetilde{Ш}_{s,s} \subseteq \widetilde{Ш}_{s,t}$ and, by the same argument as in Lemma 6.1 (iii), the canonical surjective map $H^1_{\mathscr{F}_s}(k, \mathcal{T}_s) \to H^1_{\mathcal{F}_s}(k, \mathcal{T}_s)$ induces an isomorphism $Ш_{s,t}/Ш_{s,s} \cong \widetilde{Ш}_{s,t}/\widetilde{Ш}_{s,s}$. In addition, Lemma 6.1 (i) implies $\widetilde{Ш}_{s,t} \subseteq H^1(k_t(\mathcal{T}_t)/k, \mathcal{T}_s)$, and Lemma 6.1 (ii) shows that $\widetilde{Ш}_{s,t} \cap \mathfrak{X}_s = \widetilde{Ш}_{s,s}$. Hence, setting $\Delta_{s,t} := \mathrm{Gal}(k_t(\mathcal{T}_t)/k_s(\mathcal{T}_s))$, there exists a composite injective map of $\mathcal{R}_s$-modules

$$Ш_{s,t}/Ш_{s,s} \cong \widetilde{Ш}_{s,t}/\widetilde{Ш}_{s,s} \hookrightarrow H^1(\Delta_{s,t}, \mathcal{T}_s) = \mathrm{Hom}(\Delta_{s,t}, \mathcal{T}_s)$$

and thus, upon taking duals, a surjective map of $\mathbb{K}$-modules

$$\mathrm{Hom}(\Delta_{s,t}, \mathcal{T}_s)^* \otimes_{\mathcal{R}_s} \mathbb{K} \twoheadrightarrow (Ш_{s,t}/Ш_s)^* \otimes_{\mathcal{R}_s} \mathbb{K}. \tag{6.35}$$

On the other hand, for $s \geq i$, there exists an exact commutative diagram

$$\begin{array}{ccccccc} 0 \to & Ш_{i,j(s)} & \longrightarrow & H^1_{\mathscr{F}_i}(k, \mathcal{T}_i) & \longrightarrow & \bigoplus_{\mathfrak{q} \in \mathcal{Q}_{j(s)}} H^1_f(k_\mathfrak{q}, \mathcal{T}_i) \\ & \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ 0 \to & Ш_s[\mathfrak{a}_i] & \longrightarrow & H^1_{\mathscr{F}_s}(k, \mathcal{T}_s)[\mathfrak{a}_i] & \longrightarrow & \bigoplus_{\mathfrak{q} \in \mathcal{Q}_{j(s)}} H^1_f(k_\mathfrak{q}, \mathcal{T}_s)[\mathfrak{a}_i] \end{array}$$

analogous to that following (6.32), and hence an isomorphism $Ш_{i,j(s)} \cong Ш_s[\mathfrak{a}_i]$ of $\mathcal{R}_i$-modules. Lemma 2.8 (iii) then gives an isomorphism $Ш_s^* \otimes_{\mathcal{R}_s} \mathcal{R}_i \cong (Ш_{i,j(s)})^*$ and hence allows us to apply Lemma 2.32 (i) with each S_n taken to be $\mathbb{K}$ (so that the target module, and hence cokernel, of the map (2.33) vanishes) to conclude the induced map $\mathrm{Tor}_1^{\mathcal{R}_s}(Ш_s^*, \mathbb{K}) \to \mathrm{Tor}_1^{\mathcal{R}_i}((Ш_{i,j(s)})^*, \mathbb{K})$ is surjective. This in turn gives rise to an exact commutative diagram

$$\begin{array}{ccccc} \mathrm{Tor}_1^{\mathcal{R}_i}(\mathbb{K}, (Ш_{i,j(s)})^*)^* & \longrightarrow & \mathrm{Tor}_1^{\mathcal{R}_i}(\mathbb{K}, Ш_i^*)^* & \longrightarrow & \mathbb{K} \otimes_{\mathcal{R}_i} (Ш_{i,j(s)}/Ш_i)^* \\ \downarrow & & \downarrow & & \\ \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, Ш_s^*)^* & \xrightarrow{\mathrm{id}} & \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, Ш_s^*)^*. & & \end{array}$$

By applying the Snake Lemma to this diagram and then taking direct limits over $s \geq i$, we obtain a composite injective homomorphism

$$\ker(\Theta_i) \cong \varinjlim_{s \geq i} \ker\big(\mathrm{Tor}_1^{\mathcal{R}_i}(\mathbb{K}, Ш_i^*)^* \to \mathrm{Tor}_1^{\mathcal{R}_s}(\mathbb{K}, Ш_s^*)^*\big) \hookrightarrow \varinjlim_{s \geq i}\big(\mathbb{K} \otimes_{\mathcal{R}_i} (Ш_{i,j(s)}/Ш_i)^*\big). \tag{6.36}$$

In addition, the natural isomorphisms for $s \geq i$

$$\mathrm{Hom}(\Delta_{i,j(s)}, \mathcal{T}_i)^* \otimes_{\mathcal{R}_i} \mathbb{K} \cong \big(\mathrm{Hom}(\Delta_{i,j(s)}, \mathcal{T}_i)[\mathcal{M}_i]\big)^\vee \cong \mathrm{Hom}(\Delta_{i,j(s)}, \overline{\mathcal{T}})^\vee$$

combine to give an isomorphism

$$\varinjlim_{s \geq i} \mathrm{Hom}(\Delta_{i,j(s)}, \mathcal{T}_i)^* \otimes_{\mathcal{R}_i} \mathbb{K} \cong (\varprojlim_{s \geq i} \mathrm{Hom}(\Delta_{i,j(s)}, \overline{\mathcal{T}}))^\vee \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k_\infty(\mathcal{T})/k_i(\mathcal{T}_i)), \overline{\mathcal{T}})^\vee$$

and hence also, in conjunction with (6.35), an induced surjective map

$$\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k_\infty(\mathcal{T})/k_i(\mathcal{T}_i)), \overline{\mathcal{T}})^\vee \twoheadrightarrow \varinjlim_{s \geq i}\big(\mathbb{K} \otimes_{\mathcal{R}_i} (Ш_{i,j(s)}/Ш_i)^*\big).$$

This combines with the injective map (6.36) to imply that

$$\begin{aligned}\dim_{\mathbb{K}}\big(\ker(\Theta_i)\big) &\le \dim_{\mathbb{K}}\big(\varinjlim_{s\ge i}\big(\mathbb{K}\otimes_{\mathcal{R}_i}(\text{Ш}_{i,j(s)}/\text{Ш}_i)^*\big)\big)\\ &\le \dim_{\mathbb{K}}\big(\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k_\infty(\mathcal{T})/k_i(\mathcal{T}_i)),\overline{\mathcal{T}})^\vee\big)\\ &= \dim_{\mathbb{F}_p}\big(\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k(\mathcal{T})_\infty/k_i(\mathcal{T}_i)),\mathbb{F}_p)\big)\cdot\dim_{\mathbb{K}}(\overline{\mathcal{T}})\\ &= \dim_{\mathbb{F}_p}\big(\mathrm{Gal}(k(\mathcal{T})_\infty/k_i(\mathcal{T}_i))^{\mathrm{ab}}\otimes_{\mathbb{Z}_p}\mathbb{F}_p\big)\cdot\dim_{\mathbb{K}}(\overline{\mathcal{T}})\\ &= \mathrm{d}\big(\mathrm{Gal}(k(\mathcal{T})_\infty/k_i(\mathcal{T}_i))\big)\cdot\dim_{\mathbb{K}}(\overline{\mathcal{T}})\\ &\le \mathrm{rk}(k(\mathcal{T})_\infty/k)\cdot\dim_{\mathbb{K}}(\overline{\mathcal{T}}).\end{aligned}$$

Here we write $\mathrm{d}\big(\mathrm{Gal}(k(\mathcal{T})_\infty/k_i(\mathcal{T}_i))\big)$ for the minimal number of topological generators of the pro-p group $\mathrm{Gal}(k(\mathcal{T})_\infty/k_i(\mathcal{T}_i))$, so that the second equality follows from a standard property of the Frattini subgroup (cf. [32, Prop. 1.9, Prop. 1.13]) and, since $\mathrm{Gal}(k(\mathcal{T})_\infty/k_i(\mathcal{T}_i))$ is a closed subgroup of a Sylow p-subgroup of $\mathrm{Gal}(k(\mathcal{T})_\infty/k)$, the final inequality follows from [32, Prop. 3.11, Def. 3.12]. □

(6.37) Remark. The proof of Lemma 6.34 is the only point in the proof of Theorem 4.20 in which Hypothesis 4.14 (vii) is used. In particular, if the Tate-Shafarevich group $\text{Ш}_{\mathscr{F}_s,j(s)}(\mathcal{T}_s)$ vanishes for each $s\in\mathbb{N}$, then Lemma 6.34 plays no role in the proof of Theorem 6.28 and so Hypothesis 4.14 (vii) can be omitted from the statement of Theorem 4.20.

6.2.4. Consequences for Kolyvagin systems

We can now finally prove the key technical result concerning Kolyvagin systems that will be used in the next section to prove Theorem 4.20. In order to state this result, for $i\in\mathbb{N}$ we set

$$\lambda_i^*(1) := \lambda^*_{\overline{F_i}}(1),$$

where $\overline{F_i}$ is the Mazur–Rubin structure on $\overline{T}$ that is induced by $F_i = h(\mathscr{F}_i\otimes_{\mathcal{R}_i} R_i)$ and the second term in the display is the non-negative integer defined in Remark 6.4. For $i\in\mathbb{N}$ and $d\in\mathbb{N}_0$ we then define a polynomial $Z_i(d;X)$ of degree d in $\mathbb{Z}[X]$ by setting

$$Z_i(d;X) := 8\lambda_i^*(1)X^d + \sum\nolimits_{j=0}^{j=d-1} X^j.$$

(6.38) Theorem. *Assume Hypotheses 4.14 and fix a non-negative integer t for which $t_{\mathfrak{F}} := t+\chi_{\mathfrak{F}} > 0$. Then, for every $i\in\mathbb{N}$, there exists a relative core vertex $\mathfrak{n}_0$ for $\mathcal{F}_i$ that belongs to $\mathcal{N}_{j(i)}$ and also has the following property. If $\kappa = (\kappa_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}_i}$ is any system in $\mathrm{KS}^{t_{\mathfrak{F}}}(\mathscr{F}_i)$ for which $\kappa_{\mathfrak{n}_0}=0$, then there exists an ideal I of $\mathcal{R}_i$ that satisfies both of the following conditions.*

(i) The annihilator of κ_1 in $\mathcal{R}_i$ contains $I\cdot\mathrm{Fitt}^t_{\mathcal{R}_i}(X(\mathscr{F}_i))^{9\lambda_i^*(1)}$.

(ii) The R_i-module J_i in Definition 4.13 satisfies $\mathrm{Fitt}^0_{R_i}(J_i)^{Z_i(\lambda_i^*(1);\,\alpha_i(1)+\lambda_i^*(1))}\subseteq R_i\cdot\varrho_i\big(I\big)$.

Proof. Our argument will show that the required properties are satisfied by any relative core vertex for $\mathcal{F}_i$ that is constructed as in Lemma 6.12 with $j=j(i)$. Before verifying this, however, it is convenient to first prove the existence of an ideal I_1' of $\mathcal{R}_i$ that satisfies both

$$\mathrm{Fitt}^0_{R_i}(J_i)\subseteq R_i\cdot\varrho_i(I_1') \quad\text{and}\quad (I_1'\cdot\mathrm{Fitt}^t_{\mathcal{R}_i}(X(\mathscr{F}_i)))\cdot{\bigcap}^{t_{\mathfrak{F}}}_{\mathcal{R}_i}\text{Ш}_{\mathscr{F}_i,j(i)}(\mathcal{T}_i)=(0). \tag{6.39}$$

To do this, we note Lemma 2.11 (v) directly implies that ${\bigcap}^{t_{\mathfrak{F}}}_{\mathcal{R}_i}\text{Ш}_{\mathscr{F}_i,j(i)}(\mathcal{T}_i)$ is annihilated by $\mathrm{Fitt}^{t_{\mathfrak{F}}-1}_{\mathcal{R}_i}(\text{Ш}_{\mathscr{F}_i,j(i)}(\mathcal{T}_i))$. To study the latter ideal, we first use Lemma 6.12 to fix a modulus $\mathfrak{a}$ in $\mathcal{N}_{j(i)}$ such that $H^1_{F_i^*(\mathfrak{a})}(k,T_i^*(1))$, and hence also (by Remark 6.10) $H^1_{(\overline{F_i})^*(\mathfrak{a})}(k,\overline{T}^*(1))$, vanishes. We then use Proposition 6.5 to fix a prime $\mathfrak{q}\in\mathcal{Q}_{j(i)}\setminus V(\mathfrak{a})$ such that the localisation map $H^1_{\overline{F_i}(\mathfrak{a})}(k,\overline{T})\to H^1_f(k_\mathfrak{q},\overline{T})$ is nonzero (and thus surjective). Given this, the global duality long exact sequence (5.17) (with $\Phi=\tilde{\mathcal{F}}=\overline{F_i}$) implies that $H^1_{(\overline{F_{i,\mathfrak{q}}})^*(\mathfrak{a})}(k,\overline{T}^*(1)) = H^1_{(\overline{F_i})^*(\mathfrak{a})}(k,\overline{T}^*(1))$.

It follows that $H^1_{(\overline{F_{i,\mathfrak{q}}})^*(\mathfrak{a})}(k,\overline{T}^*(1))$ vanishes and hence, by Lemma 6.13, that the first property in (6.39) is satisfied by the ideal

$$I_1' := \mathrm{Fitt}^0_{\mathcal{R}_i}(H^1_{(\mathcal{F}_{i,\mathfrak{q}})^*(\mathfrak{a})}(k,\mathcal{T}_i^*(1))^*).$$

In addition, since $\text{Ш}_{\mathscr{F}_i,j(i)}(\mathcal{T}_i)$ is, by its very definition, a submodule of $H^1_{\mathscr{F}_{i,\mathfrak{q}}(\mathfrak{a})}(k,\mathcal{T}_i)$, there is a natural surjective homomorphism $H^1_{\mathscr{F}_{i,\mathfrak{q}}(\mathfrak{a})}(k,\mathcal{T}_i)^* \twoheadrightarrow \text{Ш}_{\mathscr{F}_i,j(i)}(\mathcal{T}_i)^*$. Taken together with Lemma 2.11 (i), this surjective map implies an inclusion

$$\mathrm{Fitt}^{t_{\mathfrak{F}}-1}_{\mathcal{R}_i}(H^1_{\mathscr{F}_{i,\mathfrak{q}}(\mathfrak{a})}(k,\mathcal{T}_i)^*) \subseteq \mathrm{Fitt}^{t_{\mathfrak{F}}-1}_{\mathcal{R}_i}(\text{Ш}_{\mathscr{F}_i,j(i)}(\mathcal{T}_i)^*).$$

Given this, the fact that the above ideal I_1' also has the second property in (6.39) is a direct consequence of the inclusion $I_1' \cdot \mathrm{Fitt}^t_{\mathcal{R}_i}(X(\mathscr{F}_i)) \subseteq \mathrm{Fitt}^{t_{\mathfrak{F}}-1}_{\mathcal{R}_i}(H^1_{\mathfrak{F}_{i,\mathfrak{q}}(\mathfrak{a})}(k,\mathcal{T}_i)^*)$ proved in (6.23) (with $\mathfrak{n}$ replaced by $\mathfrak{a}$).

Turning now to the proof of the claimed result, we fix a relative core vertex $\mathfrak{n}_0$ for $\mathcal{F}_i$ that is constructed as in Lemma 6.12 with $j = j(i)$ (so that $\mathfrak{n}_0 \in \mathcal{N}_{j(i)}$ and $\nu(\mathfrak{n}_0) = \lambda_i^*(1)$). We also assume to be given a Kolyvagin system κ in $\mathrm{KS}^{t_{\mathfrak{F}}}(\mathscr{F}_i)$ for which $\kappa_{\mathfrak{n}_0} = 0$. It is then enough for us to prove the following: for any modulus $\mathfrak{n}$ in $\mathcal{N}_{j(i)}$ that satisfies

$$\lambda_i^*(\mathfrak{n}) = \lambda_i^*(1) - \nu(\mathfrak{n}), \tag{6.40}$$

there exists an ideal $I_\mathfrak{n}$ of $\mathcal{R}_i$ that satisfies both

$$\mathrm{Fitt}^0_{R_i}(J_i)^{Z_i(\lambda_i^*(\mathfrak{n}))} \subseteq R_i \cdot \varrho_i(I_\mathfrak{n}) \quad \text{and} \quad I_\mathfrak{n} \cdot \mathrm{Fitt}^t_{\mathcal{R}_i}(X(\mathscr{F}_i))^{\lambda_i^*(\mathfrak{n})+8\lambda_i^*(1)} \cdot \kappa_\mathfrak{n} = \{0\}, \tag{6.41}$$

where, for brevity, we have set

$$Z_i(\lambda_i^*(\mathfrak{n})) := Z_i(\lambda_i^*(\mathfrak{n});\, \alpha_i(1) + \lambda_i^*(1)).$$

Indeed, since $\mathfrak{n} = 1$ satisfies (6.40), if this claim is valid, then we will obtain an ideal I of $\mathcal{R}_i$ with all of the required properties by simply setting $I := I_1$.

Now, to construct ideals $I_\mathfrak{n}$ satisfying (6.41), we shall use an induction on the non-negative integer $\lambda_i^*(\mathfrak{n})$. We therefore first assume that $\lambda_i^*(\mathfrak{n}) = 0$, so that the exponents $Z_i(\lambda_i^*(\mathfrak{n}))$ and $\lambda_i^*(\mathfrak{n}) + 8\lambda_i^*(1)$ in (6.41) are both equal to $8\lambda_i^*(1)$. Then, in this case, Remark 6.10 implies $\mathfrak{n}$ is a relative core vertex for $\mathcal{F}_i$ and (6.40) implies $\nu(\mathfrak{n}) = \lambda_i^*(1)$. Upon combining Proposition 6.5 with $n = i$ and $j = j(i)$ with the argument of Proposition 6.21 with $\mathfrak{n}_1$ and $\mathfrak{n}_2$ taken to be $\mathfrak{n}$ and $\mathfrak{n}_0$, we can therefore deduce that $\mathfrak{n}$ is connected to $\mathfrak{n}_0$ in $\mathcal{X}^0_{j(i)}(i)$ by a path U of length at most $8\lambda_i^*(1)$. Given this, and the assumed vanishing of $\kappa_{\mathfrak{n}_0}$, the results of Lemmas 6.13 and 6.24 directly imply that (6.41) is validated by the ideal

$$I_\mathfrak{n} := \prod\nolimits_{(\mathfrak{q},\mathfrak{m})\in U} \mathrm{Fitt}^0_{\mathcal{R}_i}(H^1_{(\mathcal{F}_{i,\mathfrak{q}})^*(\mathfrak{m})}(k,\mathcal{T}_i^*(1))^*).$$

For the inductive step we now assume to be given a modulus $\mathfrak{n}$ in $\mathcal{N}_{j(i)}$ that satisfies (6.40) and is also such that both $\lambda_i^*(\mathfrak{n}) > 0$ and ideals $I_\mathfrak{m}$ with the properties in (6.41) are known to exist for all moduli $\mathfrak{m}$ in $\mathcal{N}_{j(i)}$ that satisfy both (6.40) and $\lambda_i^*(\mathfrak{m}) < \lambda_i^*(\mathfrak{n})$.

In this case $H^1_{\overline{F_i}^*(\mathfrak{n})}(k,\overline{T}^*(1))$ is non-trivial as $\lambda_i^*(\mathfrak{n}) > 0$, and hence, since $\chi(F_i, j(i)) \geq 0$ (by Hypothesis 4.14 (vi)), one also has $H^1_{\overline{F_i}(\mathfrak{n})}(k,\overline{T}) \neq \text{Ш}_{\overline{F_i},j(i)}(\overline{T})$. In particular, after recalling the definition (6.27) of $\alpha_i(\mathfrak{n})$, Lemma 6.26 can be combined with Proposition 6.5 (ii) to deduce the existence of a subset $\{\mathfrak{q}_l : l \in [\alpha_i(\mathfrak{n})]\}$ of $\mathcal{Q}_{j(i)}$ for which the localisation map

$$H^1_{\mathscr{F}_i(\mathfrak{n})}(k,\mathcal{T}_i)/\text{Ш}_{\mathscr{F}_i,j(i)}(\mathcal{T}_i) \to \bigoplus\nolimits_{l\in[\alpha_i(\mathfrak{n})]} H^1_f(k_{\mathfrak{q}_l},\mathcal{T}_i) \tag{6.42}$$

is injective and, simultaneously, for every $l \in [\alpha_i(\mathfrak{n})]$, the localisation maps

$$H^1_{\overline{F_i}(\mathfrak{n})}(k,\overline{T}) \to H^1(k_{\mathfrak{q}_l},\overline{T}) \quad \text{and} \quad H^1_{\overline{F_i}^*(\mathfrak{n})}(k,\overline{T}^*(1)) \to H^1(k_{\mathfrak{q}_l},\overline{T}^*(1))$$

are both nonzero. For every $l \in [\alpha_i(\mathfrak{n})]$, one then has

$$\lambda_i^*(\mathfrak{n}\mathfrak{q}_l) = \lambda_i^*(\mathfrak{n}) - 1 = \lambda_i^*(1) - \nu(\mathfrak{n}) - 1 = \lambda_i^*(1) - \nu(\mathfrak{n}\mathfrak{q}_l),$$

where the first equality follows from [26, Prop. 5.7], the second from (6.40) and the last is clear. It follows that the modulus $\mathfrak{n}\mathfrak{q}_l$ satisfies both (6.40) and also $\lambda_i^*(\mathfrak{n}\mathfrak{q}_l) < \lambda_i^*(\mathfrak{n})$ and so the inductive hypothesis implies the existence of an ideal $I_{\mathfrak{n}\mathfrak{q}_l}$ of $\mathcal{R}_i$ that has the properties in (6.41) after replacing $\mathfrak{n}$ by $\mathfrak{n}\mathfrak{q}_l$.

In particular, since $\lambda_i^*(\mathfrak{n}\mathfrak{q}_l) = \lambda_i^*(\mathfrak{n}) - 1$, the ideal $I_{\mathfrak{n}\mathfrak{q}_l} \cdot \mathrm{Fitt}^t_{\mathcal{R}_i}(X(\mathscr{F}_i))^{(\lambda_i^*(\mathfrak{n})+8\lambda_i^*(1))-1}$ is known to annihilate the element $\mathrm{loc}_{\mathfrak{q}_l}(\kappa_{\mathfrak{n}\mathfrak{q}_l})$ of $\bigcap^{t_{\mathfrak{F}}-1}_{\mathcal{R}_i} H^1_{\mathscr{F}_{i,\mathfrak{q}_l}(\mathfrak{n})}(k,\mathcal{T}_i) \subseteq \bigcap^{t_{\mathfrak{F}}-1}_{\mathcal{R}_i} H^1_{\mathscr{F}_i(\mathfrak{n})}(k,\mathcal{T}_i)$. From the equality $\psi^{\mathrm{fs}}_{\mathfrak{q}_l}(\kappa_{\mathfrak{n}}) = v_{\mathfrak{q}_l}(\kappa_{\mathfrak{n}\mathfrak{q}_l})$ that follows from the defining relation of Kolyvagin systems, it therefore follows that $\psi^{\mathrm{fs}}_{\mathfrak{q}_l}(\kappa_{\mathfrak{n}})$ is also annihilated by $I_{\mathfrak{n}\mathfrak{q}_l} \cdot \mathrm{Fitt}^t_{\mathcal{R}_i}(X(\mathscr{F}_i))^{(\lambda_i^*(\mathfrak{n})+8\lambda_i^*(1))-1}$.

Next we note the injectivity of (6.42) implies $Ш_{\mathscr{F}_i,j(i)}(\mathcal{T}_i)$ is the kernel of the diagonal map

$$(\psi^{\mathrm{fs}}_{\mathfrak{q}_l})_{l\in[\alpha_i(\mathfrak{n})]} \colon H^1_{\mathscr{F}_i(\mathfrak{n})}(k,\mathcal{T}_i) \to \mathcal{R}^{\oplus \alpha_i(\mathfrak{n})}.$$

From Lemma 2.17 (i), it then follows that $\bigcap^{t_{\mathfrak{F}}}_{\mathcal{R}_i} Ш_{\mathscr{F}_i,j(i)}(\mathcal{T}_i)$ is the kernel of the map

$$\bigcap^{t_{\mathfrak{F}}}_{\mathcal{R}_i} H^1_{\mathscr{F}_i(\mathfrak{n})}(k,\mathcal{T}_i) \xrightarrow{(\psi^{\mathrm{fs}}_{\mathfrak{q}_l})_{l\in[\alpha_i(\mathfrak{n})]}} \bigoplus_{l\in[\alpha_i(\mathfrak{n})]} \bigcap^{t_{\mathfrak{F}}-1}_{\mathcal{R}_i} H^1_{\mathscr{F}_i(\mathfrak{n})}(k,\mathcal{T}_i),$$

and so the above observations imply an inclusion

$$\Big(\prod_{l\in[\alpha_i(\mathfrak{n})]} I_{\mathfrak{n}\mathfrak{q}_l}\Big) \cdot \mathrm{Fitt}^t_{\mathcal{R}_i}(X(\mathscr{F}_i))^{(\lambda_i^*(\mathfrak{n})+8\lambda_i^*(1))-1} \cdot \kappa_{\mathfrak{n}} \subseteq \bigcap^{t_{\mathfrak{F}}}_{\mathcal{R}_i} Ш_{\mathscr{F}_i,j(i)}(\mathcal{T}_i). \tag{6.43}$$

In addition, there are also inclusions

$$\begin{aligned}\mathrm{Fitt}^0_{R_i}(J_i)^{Z_i(\lambda_i^*(\mathfrak{n}))} &= \mathrm{Fitt}^0_{R_i}(J_i) \cdot \big(\mathrm{Fitt}^0_{R_i}(J_i)^{Z_i(\lambda_i^*(\mathfrak{n}\mathfrak{q}_l))}\big)^{\alpha_i(1)+\lambda_i^*(1)} \\ &\subseteq \mathrm{Fitt}^0_{R_i}(J_i) \cdot \prod_{l\in[\alpha_i(\mathfrak{n})]} \mathrm{Fitt}^0_{R_i}(J_i)^{Z_i(\lambda_i^*(\mathfrak{n}\mathfrak{q}_l))} \\ &\subseteq \mathrm{Fitt}^0_{R_i}(J_i) \cdot \varrho_i\Big(\prod_{l\in[\alpha_i(\mathfrak{n})]} I_{\mathfrak{n}\mathfrak{q}_l}\Big).\end{aligned} \tag{6.44}$$

Here the equality is valid for every $l \in [\alpha_i(\mathfrak{n})]$ and follows from the fact that $Z_i(d;X) = 1 + Z_i(d-1;X)X$ for each $d > 0$. In addition, the first inclusion follows directly from the inequalities

$$\alpha_i(\mathfrak{n}) \le \alpha_i(1) + \nu(\mathfrak{n}) = \alpha_i(1) + (\lambda_i^*(1) - \lambda_i^*(\mathfrak{n})) < \alpha_i(1) + \lambda_i^*(1),$$

where the initial inequality is obtained via (repeated applications of) Theorem 6.28 (i) and the equality is an immediate consequence of (6.40), and the second inclusion in (6.44) follows from the induction hypothesis.

Upon combining the inclusions (6.43) and (6.44) with those of (6.39), it is now easily deduced that the product ideal $I_{\mathfrak{n}} := I_1' \cdot \prod_{l\in[\alpha_i(\mathfrak{n})]} I_{\mathfrak{n}\mathfrak{q}_l}$ satisfies both conditions in (6.41), thereby verifying the inductive step. This therefore completes the proof of the claimed result. □

(6.45) Remark. We make some observations about the exponents that occur in Theorem 6.38. We recall first that $\lambda_i^*(1)$ is defined to be $\dim_{\Bbbk}\big(H^1_{F_i^*}(k,\overline{T}^\vee(1))\big)$ and hence that

$$0 \le 9\lambda_i^*(1) \le N := 9 \cdot \dim_{\Bbbk}\big(H^1(\mathcal{O}_{k,S(\mathscr{F}_k)},\overline{T}^\vee(1))\big). \tag{6.46}$$

Since this bound is independent of i it implies, in particular, that the set of polynomials $\{Z_j(\lambda_j^*(1):X) : j \in \mathbb{N}\}$ is finite. In addition, the value of any polynomial $Z_j(\lambda_j^*(1):X)$ at a non-negative integer is a natural number and so we can set

$$Z := \max\{Z_j(\lambda_j^*(1):m) : j \in \mathbb{N},\, m \in \mathbb{N}_0, m \le \Phi_{\mathscr{F}_k}(N)\},$$

where $\Phi_{\mathscr{F}_k}$ is the $\mathbb{N}_0$-valued function that occurs in Theorem 6.28 (ii). Then the latter result (with $\mathfrak{n} = 1$) combines with (6.46) to imply that, for every $i \in \mathbb{N}$, one has

$$0 < Z_i(\lambda_i^*(1);\, \alpha_i(1) + \lambda_i^*(1)) \le Z. \tag{6.47}$$

7. Stark systems and the proof of Theorem 4.20

In this section, we fix a family $\mathfrak{F} = (\mathscr{F}_K)_{K\in\Omega}$ of Nekovář systems as in §4.3 and continue to write $\mathscr{F}$ and $\mathscr{F}_i$ for each natural number i in place of $\mathscr{F}_k$ and $\mathscr{F}_{k,i}$.

We recall that the notion of 'Stark systems' was independently introduced by Mazur and Rubin in [76] and by Sano in [98]. To define a suitable variant for our theory we fix an ordering $\prec$ on Π_k and use the following sign convention. Given a modulus $\mathfrak{n} \in \mathcal{N}_i$, we label the primes in $V(\mathfrak{n})$ as $\mathfrak{q}_1, \dots, \mathfrak{q}_{\nu(\mathfrak{n})}$ in such a way that $\mathfrak{q}_1 \prec \mathfrak{q}_2 \prec \cdots \prec \mathfrak{q}_{\nu(\mathfrak{n})}$. For any set $\{a_\mathfrak{q} : \mathfrak{q} \in V(\mathfrak{n})\}$ indexed by $V(\mathfrak{n})$, we then define the exterior product $\bigwedge_{\mathfrak{q}\in V(\mathfrak{n})} a_\mathfrak{q}$ to be $\bigwedge_{i\in[\nu(\mathfrak{n})]} a_{\mathfrak{q}_i}$. Further, if a modulus $\mathfrak{m} \in \mathcal{N}_i$ is divisible by $\mathfrak{n}$, then we define a 'sign' $\mathrm{sgn}(\mathfrak{m},\mathfrak{n}) \in \{\pm 1\}$ via the equality

$$\big(\wedge_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})} \mathfrak{q}\big) \wedge \big(\wedge_{\mathfrak{q}\in V(\mathfrak{n})} \mathfrak{q}\big) = \mathrm{sgn}(\mathfrak{m},\mathfrak{n}) \cdot \wedge_{\mathfrak{q}\in V(\mathfrak{m})}\mathfrak{q}$$

in the exterior algebra $\bigwedge^*_\mathbb{Z} \mathbb{Z}[\Pi_k]$.

(7.1) Definition. *Fix $t \in \mathbb{N}_0$ and $i \in \mathbb{N}$. Then the $\mathcal{R}_i$-module of 'Stark systems' of rank t for the Nekovář structure $\mathscr{F}_i$ on $\mathcal{T}_i$ is the inverse limit*

$$\mathrm{StS}^t(\mathscr{F}_i) := \varprojlim_{\mathfrak{n}\in\mathcal{N}_i} \bigcap\nolimits_{\mathcal{R}_i}^{t+\nu(\mathfrak{n})} H^1_{\mathscr{F}_i^\mathfrak{n}}(k,\mathcal{T}_i).$$

Here the transition morphisms are the maps

$$v_{\mathfrak{m},\mathfrak{n}} := \mathrm{sgn}(\mathfrak{m},\mathfrak{n}) \cdot \wedge_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})}\hat{v}_\mathfrak{q} \colon \bigcap\nolimits_{\mathcal{R}_i}^{t+\nu(\mathfrak{m})} H^1_{\mathscr{F}_i^\mathfrak{m}}(k,\mathcal{T}_i) \to \bigcap\nolimits_{\mathcal{R}_i}^{t+\nu(\mathfrak{n})} H^1_{\mathscr{F}_i^\mathfrak{n}}(k,\mathcal{T}_i) \qquad (7.2)$$

obtained by applying Lemma 2.17 (b) to the exact sequence (5.14) with $\Phi = \mathscr{F}_i$.

The reader will find a detailed axiomatic treatment of Stark systems in the next section. For the moment, however, we shall only record an algebraic construction of such systems that is needed for the proof of Theorem 4.20. We observe that this construction is modeled on that of Sano and the second author in [27, Th. 3.4].

To state the result, we use the free $\mathcal{R}$-module Y, of rank r_Y, that is fixed at the beginning of §4.3 and, for each natural number i, write Y_i for its image under the natural projection map

$$\bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\infty} H^0(k_\mathfrak{q}, \mathcal{T}^\vee(1))^\vee \to \bigoplus\nolimits_{\mathfrak{q}\in\Pi_k^\infty} H^0(k_\mathfrak{q}, \mathcal{T}_i^*(1))^*. \qquad (7.3)$$

The $\mathcal{R}_i$-module Y_i is free of rank r_Y and, via the map α_3 in (3.46), we regard it as a quotient of $X(\mathscr{F}_i)$. We also note that, for all $\mathfrak{n}$ and $\mathfrak{m}$ in $\mathcal{N}_i$ with $\mathfrak{n} \mid \mathfrak{m}$, the exact triangle in the lower row of (5.20) (in which $\tilde{Y}, \tilde{\mathscr{F}}$ and $\mathcal{A}$ correspond to $Y_i, \mathscr{F}_i$ and $\mathcal{T}_i$) combines with the identification, for each $\mathfrak{q} \in V(\mathfrak{m}/\mathfrak{n})$, of $H^1_{/f}(k_\mathfrak{q},\mathcal{T}_i)$ with $\mathcal{R}_i$ that is used in the construction of the upper square of the diagram in Proposition 5.22 (iii) (and earlier in the derivation of the duality sequence (5.14)) to induce an isomorphism of $\mathcal{R}_i$-modules

$$\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i^\mathfrak{m})) \cong \mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i^\mathfrak{n})) \otimes_{\mathcal{R}_i} \bigotimes\nolimits_{\mathfrak{q}\in V(\mathfrak{m}/\mathfrak{n})} \mathrm{Det}_{\mathcal{R}_i}(H^1_{/f}(k_\mathfrak{q},\mathcal{T}_i)[0]) \cong \mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i^\mathfrak{n})).$$

Finally, we fix an (ordered) $\mathcal{R}$-basis $b_\bullet$ of Y and write $b_{\bullet,i}$ for the (ordered) $\mathcal{R}_i$-basis of Y_i given by the image of $b_\bullet$ under (7.3). Then, to ensure compatibility of our constructions under change of i, we assume that the map $\vartheta_{\mathscr{F}_i^\mathfrak{n},Y_i}$ that occurs in Proposition 5.22 (iii) is defined by using the basis $b_{\bullet,i}$.

(7.4) Lemma. *Set $r := r_Y + \chi_\mathcal{R}(C(\mathscr{F}_k))$ and assume $r > 0$. Then, for each $i \in \mathbb{N}$, the assignment $(a_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}_i} \mapsto (\vartheta_{\mathscr{F}_i^\mathfrak{n},Y_i}(a_\mathfrak{n}))_{\mathfrak{n}\in\mathcal{N}_i}$ induces a well-defined homomorphism of $\mathcal{R}_i$-modules*

$$\varprojlim_{\mathfrak{n}\in\mathcal{N}_i} \mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i^\mathfrak{n})) \to \mathrm{StS}^r(\mathscr{F}_i),$$

in which the inverse limit is defined with respect to the morphisms specified above.

Proof. This follows directly from the commutativity of the upper square in the diagram of Proposition 5.22 (iii). □

In order to prove Theorem 4.20, we now fix an Euler system c in $\mathrm{ES}^r(\mathfrak{F})$. For $i \in \mathbb{N}$, we write $\kappa_i = (\kappa_{i,\mathfrak{n}})_{\mathfrak{n}\in\mathcal{N}_i}$ for the Kolyvagin system in $\mathrm{KS}^r(\mathscr{F}_i)$ that is given by the Kolyvagin derivative of c. We recall, in particular, that

$$\kappa_{i,1} = \pi^r_{k,\mathscr{F}_i}(c_k), \tag{7.5}$$

where $\pi^r_{k,\mathscr{F}_i}$ is the canonical map $\bigcap^r_{\mathcal{R}} H^1_{\mathfrak{F}}(k,\mathcal{T}) \to \bigcap^r_{\mathcal{R}_i} H^1_{\mathscr{F}_i}(k,\mathcal{T}_i)$ from (5.24).

Now, by applying Proposition 2.12 (ii) to the isomorphism $X(\mathscr{F}_k) \cong \varprojlim_i X_S(\mathscr{F}_i)$ constructed in Lemma 3.48 (ii), we deduce that the ideals $(\mathrm{Fitt}^{r_Y}_{\mathcal{R}_i}(X(\mathscr{F}_i)))_{i\in\mathbb{N}}$ form a projective system with respect to the natural maps $\mathcal{R}_{i+1} \to \mathcal{R}_i$ and also that their limit is equal to

$$\mathrm{Fitt}^{r_Y}_{\mathcal{R}}(X(\mathscr{F}_k)) = \varprojlim_{i\in\mathbb{N}} \mathrm{Fitt}^{r_Y}_{\mathcal{R}_i}(X(\mathscr{F}_i)) \subseteq \mathcal{R}.$$

In particular, if we fix an element x of $\mathrm{Fitt}^{r_Y}_{\mathcal{R}}(X(\mathscr{F}_k))$ as in the statement of Theorem 4.20, then its image under the projection map $\mathcal{R} \to \mathcal{R}_i$ belongs to $\mathrm{Fitt}^{r_Y}_{\mathcal{R}_i}(X(\mathscr{F}_i))$. As a consequence, if we write $\mathfrak{n}_i$ for the relative core vertex for $\mathcal{F}_i = h(\mathscr{F}_i)$ in $\mathcal{N}_{j(i)}$ that is provided by Lemma 6.12 and fix an element z of $\mathrm{Fitt}^0_{\mathcal{R}_i}(H^1_{\mathcal{F}_i^*(\mathfrak{n}_i)}(k,\mathcal{T}_i^*(1))^*)$, then the exact sequence in Proposition 5.9 (iii) (with $\mathfrak{a} = \mathfrak{b} = 1$) combines with Lemma 2.11 (iv) to imply that

$$z \cdot x \in \mathrm{Fitt}^0_{\mathcal{R}_i}(H^1_{\mathcal{F}_i^*(\mathfrak{n}_i)}(k,\mathcal{T}_i^*(1))^*) \cdot \mathrm{Fitt}^{r_Y}_{\mathcal{R}}(X(\mathscr{F}_k)) \subseteq \mathrm{Fitt}^{r_Y}_{\mathcal{R}}(H^1(C(\mathscr{F}_i(\mathfrak{n}_i)))).$$

The first displayed sequence in Proposition 5.22 (ii) implies that the latter ideal is equal to $\mathrm{Fitt}^0_{\mathcal{R}}(H^1(C_{Y_i}(\mathscr{F}_i(\mathfrak{n}_i))))$. Given this equality, the above containment therefore combines with the assertion in Proposition 5.22 (iii) regarding annihilation of the cokernel of $\vartheta_{\mathscr{F}_i(\mathfrak{n}_i),Y_i}$ to imply the existence of an element $\mathfrak{z}'_{\mathfrak{n}_i}$ of $\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i(\mathfrak{n}_i)))$ for which one has

$$(z \cdot x) \cdot \kappa_{i,\mathfrak{n}_i} = \vartheta_{\mathscr{F}_i(\mathfrak{n}_i),Y_i}(\mathfrak{z}'_{\mathfrak{n}_i}).$$

From the commutativity of the lower square (and surjectivity of φ_2) in the diagram of Proposition 5.22 (iii), we can then deduce the existence of an element $\mathfrak{z}_{\mathfrak{n}_i}$ of $\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i^{\mathfrak{n}_i}))$ for which the element $\epsilon'_{\mathfrak{n}_i} \coloneqq \vartheta_{\mathscr{F}_i^{\mathfrak{n}_i},Y_i}(\mathfrak{z}_{\mathfrak{n}_i})$ of $\bigcap^{r+\nu(\mathfrak{n}_i)}_{\mathcal{R}_i} H^1_{\mathscr{F}_i^{\mathfrak{n}_i}}(k,\mathcal{T}_i)$ satisfies

$$(\textstyle\bigwedge_{\mathfrak{q}\in V(\mathfrak{n}_i)} \hat{\psi}^{\mathrm{fs}}_{\mathfrak{q}})(\epsilon'_{\mathfrak{n}_i}) = (\bigwedge_{\mathfrak{q}\in V(\mathfrak{n}_i)} \hat{\psi}^{\mathrm{fs}}_{\mathfrak{q}})(\vartheta_{\mathscr{F}_i^{\mathfrak{n}_i},Y_i}(\mathfrak{z}_{\mathfrak{n}_i})) = \vartheta_{\mathscr{F}_i(\mathfrak{n}_i),Y_i}(\mathfrak{z}'_{\mathfrak{n}_i}) = (z\cdot x)\cdot \kappa_{i,\mathfrak{n}_i}.$$

Further, since $\mathfrak{z}_{\mathfrak{n}_i} \in \mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i^{\mathfrak{n}_i}))$, Lemma 7.4 implies the existence of a Stark system $\epsilon_i = (\epsilon_{i,\mathfrak{n}})_{\mathfrak{n}\in\mathcal{N}_i}$ in $\mathrm{StS}^r(\mathscr{F}_i)$ for which one has $\epsilon_{i,\mathfrak{n}_i} = \epsilon'_{\mathfrak{n}_i}$.

Next we note that a straightforward calculation, as in [26, § 5.2], proves the existence of a well-defined 'regulator map' of $\mathcal{R}_i$-modules

$$\mathrm{Reg}^r\colon \mathrm{StS}^r(\mathscr{F}_i) \to \mathrm{KS}^r(\mathscr{F}_i), \quad a = (a_{\mathfrak{n}})_{\mathfrak{n}\in\mathcal{N}_i} \mapsto \big((\textstyle\bigwedge_{\mathfrak{q}\in V(\mathfrak{n})} \hat{\psi}^{\mathrm{fs}}_{\mathfrak{q}})(a_{\mathfrak{n}})\big)_{\mathfrak{n}\in\mathcal{N}_i}.$$

Then, by its very construction, the element

$$\mathrm{Reg}^r(\epsilon_i) - (x\cdot z)\cdot\kappa_i \in \mathrm{KS}^r(\mathscr{F}_i)$$

vanishes when evaluated at $\mathfrak{n}_i$. We now write I_i for the ideal I of $\mathcal{R}_i$ that is constructed in Theorem 6.38 (with $t = r$). By combining claim (i) of the latter result with the inequality (6.46) we deduce that, for any element y of $\mathrm{Fitt}^{r_Y}_{\mathcal{R}}(X(\mathscr{F}_k))$ fixed as in the statement of Theorem 4.20, every element in the $\mathcal{R}_i$-ideal $y^N \cdot I_i$ annihilates the value

$$(\mathrm{Reg}^r(\epsilon_i) - (x\cdot z)\cdot\kappa_i)_1 = \epsilon_{i,1} - (x\cdot z)\cdot\kappa_{i,1}$$

of $\mathrm{Reg}^r(\epsilon_i) - (x\cdot z)\cdot\kappa_i$ at 1. For every element a of I_i, one therefore has an equality

$$ay^N(x\cdot z)\cdot\kappa_{i,1} = ay^N\cdot\epsilon_{i,1}. \tag{7.6}$$

On the other hand, the construction of Stark systems in Lemma 7.4 implies that

$$\epsilon_{i,1} = \vartheta_{\mathscr{F}_i,Y_i}(\mathfrak{z}''),$$

where $\mathfrak{z}''$ is the element of $\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i))$ given by the image of $\mathfrak{z}_{\mathfrak{n}_i}$ under the isomorphism φ_1 in the diagram of Proposition 5.22 (iii) with $(\mathfrak{m},\mathfrak{n})$ taken to be $(\mathfrak{n}_i, 1)$. From the equality (7.6), we can therefore derive a containment

$$az\cdot xy^N\cdot\kappa_{i,1} = y^N\cdot\vartheta_{\mathscr{F}_i,Y_i}(a\cdot\mathfrak{z}'') \in y^N\cdot\vartheta_{\mathscr{F}_i,Y_i}(\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i))).$$

Now, as the elements a and z vary, they generate the ideal $I_i' := I_i \cdot \mathrm{Fitt}^0_{\mathcal{R}_i}(H^1_{\mathcal{F}_i^*(\mathfrak{n}_i)}(k, \mathcal{T}_i^*(1))^*)$ of $\mathcal{R}_i$. From the above containment, we can therefore deduce that the following ideal of $\mathcal{R}_i$

$$\mathfrak{A}_i(x,y) := \big\{ w \in \mathcal{R}_i \mid w \cdot xy^N \cdot \kappa_{i,1} \in y^N \cdot \vartheta_{\mathscr{F}_i, Y_i}(\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i))) \big\}$$

contains I_i'. In particular, since Remark 6.15 combines with Theorem 6.38 (ii) and the inequality (6.47) to imply that $R_i \cdot \varrho_i(I_i')$ contains $\mathrm{Fitt}^0_{R_i}(J_i)^{Z+1}$, there is an inclusion

$$\mathrm{Fitt}^0_{R_i}(J_i)^{Z+1} \subseteq R_i \cdot \varrho_i\big(\mathfrak{A}_i(x,y)\big). \tag{7.7}$$

We next claim that the natural maps $\mathcal{R}_{i+1} \to \mathcal{R}_i$ send $\mathfrak{A}_{i+1}(x,y)$ to $\mathfrak{A}_i(x,y)$. To justify this, we note that the isomorphism $C(\mathscr{F}_{i+1}) \otimes^{\mathbb{L}}_{\mathcal{R}_{i+1}} \mathcal{R}_i \cong C(\mathscr{F}_i)$ in $D^{\mathrm{perf}}(\mathcal{R}_i)$ given by Proposition 5.9 (ii) and the obvious isomorphism of $\mathcal{R}_i$-modules $Y_{i+1} \otimes_{\mathcal{R}_{i+1}} \mathcal{R}_i \cong Y_i$ combine with the exact triangle given by the first column in (5.21) (with $\mathfrak{m} = 1$) to induce an isomorphism in $D^{\mathrm{perf}}(\mathcal{R}_i)$

$$\tau_i \colon C_{Y_{i+1}}(\mathscr{F}_{i+1}) \otimes^{\mathbb{L}}_{\mathcal{R}_{i+1}} \mathcal{R}_i \cong C_{Y_i}(\mathscr{F}_i).$$

This isomorphism then combines with the explicit construction of the maps $\vartheta_{\mathscr{F}_i, Y_i}$ in Proposition 5.22 (iii) (and our compatible family of bases $(b_{\bullet,i})_i$) to give a canonical commutative diagram of $\mathcal{R}_{i+1}$-modules

$$\begin{array}{ccc}
\mathrm{Det}_{\mathcal{R}_{i+1}}(C_{Y_{i+1}}(\mathscr{F}_{i+1})) & \xrightarrow{\vartheta_{\mathscr{F}_{i+1}, Y_{i+1}}} & \bigcap^r_{\mathcal{R}_{i+1}} H^1_{\mathscr{F}_{i+1}}(k, \mathcal{T}_{i+1}) \\
\big\downarrow {\scriptstyle \mathrm{Det}_{\mathcal{R}_i}(\tau_i)} & & \big\downarrow {\scriptstyle \pi^r_{i+1/i}} \\
\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i)) & \xrightarrow{\vartheta_{\mathscr{F}_i, Y_i}} & \bigcap^r_{\mathcal{R}_i} H^1_{\mathscr{F}_i}(k, \mathcal{T}_i).
\end{array}$$

where the map $\pi^r_{i+1/i}$ is constructed (via Lemma 2.37 (ii)) in the same way as the projection $\pi^r_{k,\mathscr{F}_i}$ in (5.24). In particular, from the equality (7.5) one deduces that

$$\pi^r_{i+1/i}(\kappa_{i+1,1}) = \pi^r_{i+1/i}(\pi^r_{k,\mathscr{F}_{i+1}}(c_k)) = \pi^r_{k,\mathscr{F}_i}(c_k) = \kappa_{i,1}.$$

Hence, if a belongs to $\mathfrak{A}_{i+1}(x,y)$, so that

$$a \cdot xy^N \cdot \kappa_{i+1,1} \in y^N \cdot \vartheta_{\mathscr{F}_{i+1}, Y_{i+1}}(\mathrm{Det}_{\mathcal{R}_{i+1}}(C_{Y_{i+1}}(\mathscr{F}_{i+1}))),$$

then the commutativity of the above diagram implies that

$$a \cdot xy^N \cdot \kappa_{i,1} \in y^N \cdot \vartheta_{\mathscr{F}_i, Y_i}(\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i))).$$

It follows that the image of a under the projection $\mathcal{R}_{i+1} \to \mathcal{R}_i$ belongs to $\mathfrak{A}_i(x,y)$, as claimed. In particular, since the fixed $\mathcal{R}$-basis $b_\bullet$ of Y induces an identification

$$\vartheta_{\mathscr{F}_k, Y}(\mathrm{Det}_{\mathcal{R}}(C(\mathscr{F}_k))) = \varprojlim_{i \in \mathbb{N}} \vartheta_{\mathscr{F}_i, Y_i}(\mathrm{Det}_{\mathcal{R}_i}(C_{Y_i}(\mathscr{F}_i))),$$

one obtains an inclusion

$$\varprojlim_{i \in \mathbb{N}} \mathfrak{A}_i(x,y) \subseteq \mathfrak{A}(x,y) := \big\{ r \in \mathcal{R} \mid r \cdot xy^N \cdot c_k \in y^N \cdot \vartheta_{\mathscr{F}_k, Y}(\mathrm{Det}_{\mathcal{R}}(C(\mathscr{F}_k))) \big\}.$$

Next we note that the compactness of R implies

$$\varrho\big(\varprojlim_{i \in \mathbb{N}} \mathfrak{A}_i(x,y)\big) = \varprojlim_{i \in \mathbb{N}} \varrho_i\big(\mathfrak{A}_i(x,y)\big) \subseteq R.$$

Now, by (7.7), the second limit in this display contains $(\varprojlim_i \mathrm{Fitt}^0_{R_i}(J_i))^{Z+1}$. In addition, since the canonical isomorphism (4.12) combines with the definition of the modules J_i to give an isomorphism $\mathrm{Tor}_1^{\mathcal{R}}(X(\mathscr{F}_k), R) \cong \varprojlim_{i \in \mathbb{N}} J_i$, the general result of Proposition 2.12 (ii) implies $\mathrm{Fitt}^0_R(\mathrm{Tor}_1^{\mathcal{R}}(X(\mathscr{F}_k), R))$ is equal to $\varprojlim_{i \in \mathbb{N}} \mathrm{Fitt}^0_{R_i}(J_i)$. These observations therefore combine to prove an inclusion

$$\mathrm{Fitt}^0_R(\mathrm{Tor}_1^{\mathcal{R}}(X(\mathscr{F}_k), R))^{Z+1} \subseteq R \cdot \varrho\big(\mathfrak{A}(x,y)\big).$$

At this point, we fix a prime ideal $\mathfrak{p}$ of $\mathcal{R}$ that satisfies the conditions (i) and (ii) in Theorem 4.20. Then condition (ii) combines with the above displayed inclusion to imply that $R_\mathfrak{p} \cdot \varrho(\mathfrak{A}(x,y))_\mathfrak{p}$ contains the element 1 of $R_\mathfrak{p}$. In particular, if $\varrho_\mathfrak{p} \colon \mathcal{R}_\mathfrak{p} \to R_\mathfrak{p}$ is surjective, as follows from the condition (i), then $R_\mathfrak{p} \cdot \varrho(\mathfrak{A}(x,y))_\mathfrak{p} = \varrho_\mathfrak{p}(\mathfrak{A}(x,y)_\mathfrak{p})$ and so the $\mathcal{R}_\mathfrak{p}$-ideal $\mathfrak{A}(x,y)_\mathfrak{p}$ must contain a preimage ξ of 1 under $\varrho_{\mathfrak{p}}$. In addition, condition (i) also ensures that $\varrho_{\mathfrak{p}}$ is non-trivial, so that $\ker(\varrho_{\mathfrak{p}}) \subseteq \mathfrak{p}\mathcal{R}_{\mathfrak{p}}$, and $\varrho_{\mathfrak{p}}(1) = 1$. It follows that $\xi - 1 \in \mathfrak{p}\mathcal{R}_{\mathfrak{p}}$ and so Nakayama's Lemma implies that ξ is a unit in $\mathcal{R}_{\mathfrak{p}}$. This in turn shows that $\mathfrak{A}(x,y)_{\mathfrak{p}} = \mathcal{R}_{\mathfrak{p}}$, as required to complete the proof of Theorem 4.20. □

A. Abstract Stark systems

In this appendix to Part I, we establish several useful results in the theory of Stark systems. Whilst some of these results have already been applied in previous sections, with future applications in mind we have preferred to adopt a more axiomatic approach here.

A.1. Characteristic ideals

We first recall a notion introduced by Greither [45, § 5.2] and Sakamoto [95, Def. C.3]. For this, we let $\mathcal{R}$ be a general commutative ring, take Z to be a finitely generated $\mathcal{R}$-module, and choose a surjective map of $\mathcal{R}$-modules $g\colon \mathcal{R}^{\oplus s} \twoheadrightarrow Z$. We then note Lemma 2.17 (ii) implies that the tautological short exact sequence

$$\ker(g) \xhookrightarrow{(f_i)_{i\in[s]}} \mathcal{R}^{\oplus s} \overset{g}{\twoheadrightarrow} Z$$

induces a homomorphism of $\mathcal{R}$-modules $\wedge_{i\in[s]} f_i\colon \bigcap^s_{\mathcal{R}} \ker(g) \to \bigcap^0_{\mathcal{R}}(0) = \mathcal{R}$.

(A.1) Definition. *The 'characteristic ideal'* $\mathrm{char}_{\mathcal{R}}(Z)$ *of* Z *is the image of* $\wedge_{i\in[s]} f_i$.

(A.2) Remark. Sakamoto has proved the following general results on characteristic ideals.

(a) $\mathrm{char}_{\mathcal{R}}(Z)$ does not depend on the chosen surjection $g\colon \mathcal{R}^{\oplus s} \to Z$ (see [95, Rem. C.5]).

(b) $\mathrm{Fitt}^0_{\mathcal{R}}(Z) \subseteq \mathrm{char}_{\mathcal{R}}(Z)$, with equality if $\mathrm{pd}_{\mathcal{R}}(Z) \leq 1$ (cf. [95, Prop. C.7]).

(c) $\mathrm{char}_{\mathcal{R}}(Z) = \mathrm{Ann}_{\mathcal{R}}(Z)$ if $\mathcal{R}$ is Gorenstein of dimension zero (see [94, Prop. 4.5]).

We will also use the following properties of characteristic ideals. In the sequel, for a finite group Δ we set $N_\Delta \coloneqq \sum_{\delta\in\Delta} \delta \in \mathbb{Z}[\Delta]$.

(A.3) Lemma. *Assume* $\mathcal{R}$ *is a* G_2*-ring, and let* M *be a finitely generated* $\mathcal{R}$*-module.*

(a) $\mathrm{char}_{\mathcal{R}}(M) \subseteq \mathrm{Ann}_{\mathcal{R}}(M)^{**}$.

(b) For any integers $r \geq s > 0$*, any exact sequence of* $\mathcal{R}$*-modules of the form*

$$0 \longrightarrow N \longrightarrow M \xrightarrow{(f_i)_{i\in[s]}} \mathcal{R}^{\oplus s} \longrightarrow Z \longrightarrow 0$$

and any a *in the image of the map* $\bigwedge_{s\in[r]} f_i\colon \bigcap^r_{\mathcal{R}} M \to \bigcap^{r-s}_{\mathcal{R}} N$ *in (2.19), one has*

$$\mathrm{im}(a) \coloneqq \{\varphi(a) \mid \varphi \in \bigwedge\nolimits^{r-s}_{\mathcal{R}} N^*\} \subseteq \mathrm{char}_{\mathcal{R}}(Z).$$

Moreover, if M *contains a free direct summand* F *of rank* t*, then one has*

$$\mathrm{Fitt}^t_{\mathcal{R}}(M^*) \cdot \mathrm{im}(a) \subseteq \mathrm{Fitt}^0_{\mathcal{R}}(Z)^{**}.$$

(c) For any submodule $N \subseteq M$*, one has* $\mathrm{char}_{\mathcal{R}}(M) \subseteq \mathrm{char}_{\mathcal{R}}(N) \cap \mathrm{char}_{\mathcal{R}}(M/N)$.

Proof. To prove (a), we let n be an integer that is large enough such that we can choose a surjection $f\colon \mathcal{R}^n \hookrightarrow M$. Setting $N \coloneqq \ker(f)$, we then have

$$\mathrm{char}_{\mathcal{R}}(M) \coloneqq \mathrm{im}\left\{\bigcap\nolimits^n_{\mathcal{R}} N \to \bigcap\nolimits^n_{\mathcal{R}} \mathcal{R}^{\oplus n} \cong \mathcal{R}\right\}.$$

Write $\{b_i\}_{i\in[n]}$ for the standard basis of $\mathcal{R}^n$ and, for $i \in [n]$, write $b_i^*\colon \mathcal{R}^n \to \mathcal{R}$ for the dual of b_i. We then have a commutative diagram

$$\begin{array}{ccc} \bigcap^n_{\mathcal{R}} N & \longrightarrow & \bigcap^n_{\mathcal{R}} \mathcal{R}^{\oplus n} \\ \| & & \downarrow\simeq \\ \bigcap^n_{\mathcal{R}} N & \xrightarrow{\wedge_{i\in[n]} b_i^*} & \mathcal{R}. \end{array}$$

Now, for each $i \in [n]$ the lower map factors as a composite $\bigcap_{\mathcal{R}}^{n} N \xrightarrow{\wedge_{j\neq i} b_j^*} N^{**} \xrightarrow{b_j^*} \mathcal{R}$, and so $\mathrm{char}_{\mathcal{R}}(M) \subseteq \bigcap_{i\in[n]} b_i^*(N^{**})$. Recall that the cokernel of the natural map $N \to N^{**}$ identifies with $\mathrm{Ext}^2_{\mathcal{R}}(A, \mathcal{R})$ for some $\mathcal{R}$-module A and hence, by assumption (G_1) on $\mathcal{R}$, is pseudo-null. It follows that also the cokernel of the inclusion $b_i^*(N) \subseteq b_i^*(N^{**})$ is pseudo-null. It follows that, for every prime ideal $\mathfrak{p} \in \mathrm{Spec}^{\leq 1}(\mathcal{R})$, we have

$$\mathrm{char}_{\mathcal{R}}(M)_{\mathfrak{p}} \subseteq \bigcap\nolimits_{i\in[n]} b_i^*(N)_{\mathfrak{p}} \subseteq \mathrm{Ann}_{\mathcal{R}}(M)_{\mathfrak{p}}.$$

Since reflexive ideals are uniquely determined by the localisations at prime ideals in $\mathrm{Spec}^{\leq 1}(\mathcal{R})$, we deduce that $\mathrm{char}_{\mathcal{R}}(M) \subseteq \mathrm{Ann}_{\mathcal{R}}(M)^{**}$.

To prove (b), we fix $b \in \bigcap_{\mathcal{R}}^{r} M$ and set $\theta := \wedge_{i\in[s]} f_i$. Then, for $\varphi \in \bigwedge_{\mathcal{R}}^{r-s} M^*$, one has

$$\varphi(\theta(b)) = (\theta \wedge \varphi)(b) = (-1)^{rs} \cdot (\varphi \wedge \theta)(b) = (-1)^{rs} \cdot \theta(\varphi(b)).$$

In particular, since θ is the composite $\bigcap_{\mathcal{R}}^{s} M \to \bigcap_{\mathcal{R}}^{s}(M/N) \to \bigwedge_{\mathcal{R}}^{s}(\mathcal{R}^{\oplus s}) \cong \mathcal{R}$ (where the first map is induced by the natural projection), and $\mathrm{char}_{\mathcal{R}}(Z)$ is, by definition, the image of the second map in this composite, we have shown that

$$\big\{\varphi(\theta(b)) \mid \varphi \in \bigwedge\nolimits_{\mathcal{R}}^{r-s} M^*\big\} \subseteq \mathrm{char}_{\mathcal{R}}(Z). \tag{A.4}$$

To proceed, we note $\mathrm{char}_{\mathcal{R}}(Z)$ is a reflexive ideal of $\mathcal{R}$ and hence uniquely determined by its localisations at $\mathfrak{p} \in \mathrm{Spec}^{\leq 1}(\mathcal{R})$ (cf. [95, Prop. C.12 and Lem. C.13]). To conclude the proof of (b), it is therefore sufficient to demonstrate that the first set in (A.4) is equal to $\mathrm{im}(\theta(b))$ when localised at a prime ideal in $\mathrm{Spec}^{\leq 1}(\mathcal{R})$. To do this, we use the commutative diagram

$$\begin{array}{ccc} \bigwedge_{\mathcal{R}}^{r-s} M^* & \longrightarrow & \bigwedge_{\mathcal{R}}^{r-s} N^* \\ \downarrow & & \downarrow \\ \big(\bigcap_{\mathcal{R}}^{r-s} M\big)^* & \longrightarrow & \big(\bigcap_{\mathcal{R}}^{r-s} N\big)^*. \end{array}$$

Here the upper horizontal map is induced by restriction, the vertical arrows are the respective canonical maps induced by (2.19) and the lower horizontal map is induced by the inclusion $\bigcap_{\mathcal{R}}^{r-s} N \hookrightarrow \bigcap_{\mathcal{R}}^{r-s} M$. In particular, since $\mathrm{im}(\theta) \subseteq \bigcap_{\mathcal{R}}^{r-s} N$ (by Lemma 2.17 (b)), the above diagram implies that, for $\varphi \in \bigwedge_{\mathcal{R}}^{r-s} M^*$ and $a \in \mathrm{im}(\theta)$, one can compute $\varphi(a)$ as the value at a of the image of φ in $\bigwedge_{\mathcal{R}}^{r-s} N^*$. Since, for $x \in \bigcap_{\mathcal{R}}^{r-s} N = \mathrm{Hom}_{\mathcal{R}}(\bigwedge_{\mathcal{R}}^{r-s} N^*, \mathcal{R})$ and $\psi \in \bigwedge_{\mathcal{R}}^{r-s} N^*$, the value $\psi(x) \in \mathcal{R}$ is defined to be $x(\psi)$, the above discussion therefore shows that

$$\big\{\varphi(\theta(b)) \mid \varphi \in \bigwedge\nolimits_{\mathcal{R}}^{r-s} M^*\big\} \subseteq \{\varphi(\theta(b)) \mid \varphi \in \bigwedge\nolimits_{\mathcal{R}}^{r-s} N^*\}.$$

To prove the first part of (b), it is therefore enough to show that the cokernel of this inclusion is pseudo-null. Now, for $\mathfrak{p} \in \mathrm{Spec}^{\leq 1}(\mathcal{R})$, the group $\mathrm{Ext}^1_{\mathcal{R}}\big(M/N,\ \mathcal{R}\big)_{\mathfrak{p}} = \mathrm{Ext}^2_{\mathcal{R}_{\mathfrak{p}}}(Z_{\mathfrak{p}}, \mathcal{R}_{\mathfrak{p}})$ vanishes since, by assumption, $\mathcal{R}$ satisfies (G_1) and so $\mathcal{R}_{\mathfrak{p}}$ has injective dimension one. It follows that $\mathrm{coker}\{M^* \to N^*\}_{\mathfrak{p}} = \mathrm{Ext}^1_{\mathcal{R}}(M/N, \mathcal{R})_{\mathfrak{p}}$ vanishes, and hence that the $\mathfrak{p}$-localisation of the cokernel of $\bigwedge_{\mathcal{R}}^{r-s} M^* \to \bigwedge_{\mathcal{R}}^{r-s} N^*$ also vanishes. Every $\varphi \in \bigwedge_{\mathcal{R}_{\mathfrak{p}}}^{r-s} N^*_{\mathfrak{p}}$ can therefore be lifted to an element $\psi \in \bigwedge_{\mathcal{R}_{\mathfrak{p}}}^{r-s} M^*_{\mathfrak{p}}$, and so the claimed result follows from

$$\begin{aligned}\{\varphi(\theta(b)) \mid \varphi \in \bigwedge\nolimits_{\mathcal{R}}^{r-s} N^*\}_{\mathfrak{p}} &= \{\varphi(\theta(b)) \mid \varphi \in \bigwedge\nolimits_{\mathcal{R}_{\mathfrak{p}}}^{r-s} N^*_{\mathfrak{p}}\} \\ &\subseteq \{\varphi(\theta(b)) \mid \varphi \in \bigwedge\nolimits_{\mathcal{R}_{\mathfrak{p}}}^{r-s} M^*_{\mathfrak{p}}\} = \{\varphi(\theta(b)) \mid \varphi \in \bigwedge\nolimits_{\mathcal{R}}^{r-s} M^*\}_{\mathfrak{p}}\end{aligned}$$

We now prove the second part of (b). As $\mathrm{Fitt}^0_{\mathcal{R}}(Z)^{**}$ is a reflexive ideal of $\mathcal{R}$, the discussion above shows it is enough to prove $\mathrm{Fitt}^0_{\mathcal{R}}(Z)$ contains $\mathrm{Fitt}^r_{\mathcal{R}}(M^*) \cdot \{\varphi(\theta(b)) \mid \varphi \in \bigwedge_{\mathcal{R}}^{r-s} M^*\}$.

By assumption, M has a free direct summand F of rank at least s. Then F^* is a free direct summand of M^* and the decomposition $\bigwedge_{\mathcal{R}}^{s} M^* \cong \bigoplus_{i=0}^{s}(\bigwedge_{\mathcal{R}}^{i} F^*) \otimes (\bigwedge_{\mathcal{R}}^{s-i} M^*/F^*)$ implies the kernel of the surjection $\bigwedge_{\mathcal{R}}^{s} M^* \to \bigwedge_{\mathcal{R}}^{s} F^*$ identifies with $\bigoplus_{i=0}^{s-1}(\bigwedge_{\mathcal{R}}^{i} F^*) \otimes \bigwedge_{\mathcal{R}}^{s-i}(M^*/F^*)$. Lemma 2.11 (v) now implies that $\mathrm{Fitt}^{s-i-1}_{\mathcal{R}}(M^*/F^*)$ annihilates $\bigwedge_{\mathcal{R}}^{s-i}(M^*/F^*)$ for $i \in \{0, \ldots, s-1\}$.

Since $\mathrm{Fitt}^{s-i-1}_{\mathcal{R}}(M^*/F^*)$ contains $\mathrm{Fitt}^0_{\mathcal{R}}(M^*/F^*) = \mathrm{Fitt}^{\mathrm{rk}_{\mathcal{R}}F}_{\mathcal{R}}(M^*)$, we deduce that $\mathrm{Fitt}^{\mathrm{rk}_{\mathcal{R}}F}_{\mathcal{R}}(M^*)$ annihilates the kernel of $\bigwedge^s_{\mathcal{R}} M^* \to \bigwedge^s_{\mathcal{R}} F^*$. Upon dualising, we see that $\mathrm{Fitt}^{\mathrm{rk}_{\mathcal{R}}F}_{\mathcal{R}}(M^*)$ annihilates the cokernel of the natural map $\bigwedge^s_{\mathcal{R}} F = \bigcap^r_{\mathcal{R}} F \to \bigcap^s_{\mathcal{R}} M$.

Now let $a = \varphi(b)$ for some $\varphi \in \bigwedge^{r-s}_{\mathcal{R}} M^*$. Then, for any x in $\mathrm{Fitt}^{\mathrm{rk}_{\mathcal{R}}F}_{\mathcal{R}}(M^*)$, the element $x\varphi(b)$ of $\bigcap^s_{\mathcal{R}} M$ belongs to $\bigwedge^s_{\mathcal{R}} F$. It follows that $x\varphi(a) = x(\varphi\circ\theta)(b) = (-1)^{(r-s)s}\cdot\theta(\varphi(xb))$ belongs to $\mathrm{im}\{\bigwedge^s_{\mathcal{R}} M \xrightarrow{\theta} \mathcal{R}\} = \mathrm{Fitt}^0_{\mathcal{R}}(Z)$, as claimed.

As for (c), we note Sakamoto has proved in [95, Th. C.8] that $\mathrm{char}_{\mathcal{R}}(M) \subseteq \mathrm{char}_{\mathcal{R}}(N)$. For the convenience of the reader, we provide a different argument for this inclusion in the course of the proof of (c). For this, we fix surjections $f\colon \mathcal{R}^{\oplus n} \to N$ and $g\colon \mathcal{R}^{\oplus m} \to M/N$ for large-enough integers n and m. Since $\mathcal{R}^{\oplus m}$ is projective, we can lift g to a map $\widetilde{g}\colon \mathcal{R}^{\oplus m} \to M$. The resulting map $f\oplus\widetilde{g}\colon \mathcal{R}^{\oplus(n+m)} \to M$ is then surjective, and (writing K_1, K_2, and K_3 for the relevant kernels) we obtain an exact commutative diagram

$$\begin{array}{ccccc}
K_1 & \hookrightarrow & K_3 & \twoheadrightarrow & K_2\\
\downarrow{\scriptstyle (\varphi_i)_{i\in[n]}} & & \downarrow{\scriptstyle (\varphi_i)_{i\in[n+m]}} & & \downarrow{\scriptstyle (\varphi_i)_{i\in[n+m]\setminus[n]}}\\
\mathcal{R}^{\oplus n} & \hookrightarrow & \mathcal{R}^{\oplus n}\oplus\mathcal{R}^{\oplus m} & \twoheadrightarrow & \mathcal{R}^{\oplus m}\\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\oplus\widetilde{g}} & & \downarrow{\scriptstyle g}\\
N & \hookrightarrow & M & \twoheadrightarrow & M/N
\end{array}$$

From the above diagram we can deduce the exact sequence

$$0 \to K_1 \to K_3 \xrightarrow{(\varphi_i)_{i\in[n+m]\setminus[n]}} \mathcal{R}^{\oplus m} \to M/N \to 0.$$

Now, if $a \in \bigcap^{n+m}_{\mathcal{R}} K_3$, then Lemma 2.17 (b) (applied to the above exact sequence) implies that $(\wedge_{i\in[n+m]\setminus[n]}\varphi_i)(a) \in \bigcap^n_{\mathcal{R}} K_1$, and so Lemma 2.17 (c) gives

$$\begin{aligned}
(\wedge_{i\in[n+m]}\varphi_i)(a) &= (-1)^{nm}\big((\wedge_{i\in[n]}\varphi_i)\circ(\wedge_{i\in[n+m]\setminus[n]}\varphi_i)\big)(a)\big)\\
&\in \mathrm{im}\,\big((\wedge_{i\in[n+m]\setminus[n]}\varphi_i)(a)\big) \cap (\wedge_{i\in[n]}\varphi_i)\big(\bigcap^n_{\mathcal{R}} K_1\big)\\
&\subseteq \mathrm{char}_{\mathcal{R}}(M/N) \cap (\wedge_{i\in[n]}\varphi_i)\big(\bigcap^n_{\mathcal{R}} K_1\big).
\end{aligned}$$

The claimed inclusion is then true since

$$\begin{aligned}
\mathrm{char}_{\mathcal{R}}(M) &= \mathrm{im}\,\big\{\bigcap^{n+m}_{\mathcal{R}} K_3 \to \bigwedge^{n+m}_{\mathcal{R}} \mathcal{R}^{\oplus(n+m)} \cong \mathcal{R}\big\} = (\wedge_{i\in[n+m]}\varphi_i)\big(\bigcap^{n+m}_{\mathcal{R}} K_3\big)\\
&\subseteq \mathrm{char}_{\mathcal{R}}(M/N) \cap (\wedge_{i\in[n]}\varphi_i)\big(\bigcap^n_{\mathcal{R}} K_1\big) = \mathrm{char}_{\mathcal{R}}(M/N)\cap\mathrm{char}_{\mathcal{R}}(N). \quad \square
\end{aligned}$$

A.2. The general formalism

Let $(\mathcal{Q}, \prec)$ be a totally ordered set, and regard its power set $\mathcal{P} := \mathcal{P}(\mathcal{Q})$ as a partially ordered set with respect to inclusion. If $S \in \mathcal{P}$ is a finite set, then by $\wedge_{v\in S} v$ we mean $v_1\wedge\cdots\wedge v_{|S|}$ using a labelling $S = \{v_1, \dots, v_{|S|}\}$ with $v_1 \prec \cdots \prec v_{|S|}$. If $S' \in \mathcal{P}$ is a further finite set with $S \subseteq S'$, then we define a 'sign' $\mathrm{sgn}(S', S) \in \{\pm 1\}$ via the equality

$$\big(\wedge_{v\in(S'\setminus S)} v\big) \wedge \big(\wedge_{v\in S} v\big) = \mathrm{sgn}(S', S)\cdot\wedge_{v\in S} v$$

in the exterior algebra $\bigwedge^*_{\mathbb{Z}} \mathbb{Z}[\mathcal{P}]$.

We now suppose to be given an inductive system $\mathfrak{I} := (M_S, \iota_{S,S'})_{S\in\mathcal{P}}$ of finitely generated R-modules with the following properties:

- (P_1) The maps $\iota_{S,S'}\colon M_S \to M_{S'}$ are injective (and so will be suppressed from the notation).
- (P_2) For any element v of $\mathcal{Q}$, there exists a specified map of R-modules $f_v\colon \varinjlim_S M_S \to R$, where the limit is over all elements S of $\mathcal{P}$ that contain v.
- (P_3) For all S and S' in $\mathcal{P}$ with $S \subseteq S'$, there exists a specified exact sequence of R-modules

$$0 \longrightarrow M_S \longrightarrow M_{S'} \xrightarrow{\oplus_{v\in S'\setminus S} f_v} \bigoplus_{v\in S'\setminus S} R.$$

(A.5) Definition. *For a non-negative integer r, an (abstract) 'Stark system of rank r' for $\mathfrak{I}$ is an element of the R-module*

$$\mathrm{StS}^r(\mathfrak{I}) := \varprojlim_{S\in\mathcal{P}} \bigcap\nolimits_R^{r+|S|} M_S$$

where the transition morphisms in the limit are $\mathrm{sgn}(S',S)\cdot(\bigwedge_{v\in S'\setminus S} f_v)\colon \bigcap_R^{r+|S'|} M_{S'} \to \bigcap_R^{r+|S|} M_S$ *induced by the sequence in (P$_3$) (and Lemma 2.17(b)). Such a system is therefore an element $(c_S)_{S\in\mathcal{P}}$ of $\prod_{S\in\mathcal{P}}\bigcap_R^{r+|S|} M_S$ with $\mathrm{sgn}(S',S)\cdot(\bigwedge_{v\in S'\setminus S} f_v)(c_{S'}) = c_S$ for $S\subseteq S'$.*

We now further assume the existence of an inductive system $\mathfrak{P} = \mathfrak{P}(\mathfrak{I}) := (Z_S, \rho_{S,S'})_{S\in\mathcal{P}}$ of (finitely generated) R-modules with the following properties:

(P$_4$) For $v\in\mathcal{Q}$ and $S\in\mathcal{P}$, there exists a specified map of R-modules $g_{S,v}\colon R\to Z_S$ with $\rho_{S',S}\circ g_{S,v} = g_{S',v}$ for $S\subseteq S'$.

(P$_5$) For S and S' in $\mathcal{P}$ with $S\subseteq S'$ there exists a specified exact sequence of R-modules

$$0 \longrightarrow M_S \longrightarrow M_{S'} \xrightarrow{\oplus_{v\in S'\setminus S} f_v} \bigoplus\nolimits_{v\in S'\setminus S} R \xrightarrow{\sum_{v\in S'\setminus S} g_{S,v}} Z_S \xrightarrow{\rho_{S,S'}} Z_{S'} \longrightarrow 0.$$

(A.6) Theorem. *Fix a G_2-ring R and inductive systems of R-modules $\mathfrak{I}$ and $\mathfrak{P}$ as above. Then, for each non-negative integer r, the following claims are valid.*

(i) There exist S in $\mathcal{P}$ for which $\mathrm{Fitt}_R^{r+|S|}(M_S^*)$ annihilates the kernel of the map

$$\mathrm{StS}^r(\mathfrak{I}) \to \bigcap\nolimits_R^{r+|S|} M_S, \quad (c_T)_{T\in\mathcal{P}} \mapsto c_S.$$

(ii) For every system $(c_S)_{S\in\mathcal{P}} \in \mathrm{StS}^r(\mathfrak{I})$ the ideal $\mathrm{im}(c_\varnothing)$ is contained in the characteristic ideal (over R) of the image of the map $\sum_{v\in\mathcal{Q}} g_{\varnothing,v}$ from $\bigoplus_{v\in\mathcal{Q}} R$ to $Z_\varnothing$.

Proof. Since R is noetherian, the ascending chain of ideals

$$0 = \ker(\rho_{\varnothing,\varnothing}) \subseteq \cdots \subseteq \ker(\rho_{\varnothing,S}) \subseteq \cdots$$

must terminate. This implies the existence of S in $\mathcal{P}$ with $\ker(\rho_{\varnothing,S}) = \ker(\rho_{\varnothing,S'})$ for all $S'\in\mathcal{P}$ with $S\subseteq S'$. From the exact sequence in (P$_5$), it therefore follows that $\mathrm{im}(\sum_{v\in S} g_{\varnothing,v}) = \mathrm{im}(\sum_{v\in S'} g_{\varnothing,v})$ and hence that $\ker(\rho_{\varnothing,S})$ is equal to the kernel, Ω say, of the map $\sum_{v\in\mathcal{Q}} g_{\varnothing,v}$ that occurs in (ii). One therefore has an exact sequence

$$0 \longrightarrow M_\varnothing \longrightarrow M_S \xrightarrow{\oplus_{v\in S} f_v} \bigoplus\nolimits_{v\in S} R \xrightarrow{\sum_{v\in S} g_{\varnothing,v}} \Omega \longrightarrow 0.$$

Claim (ii) follows by applying Lemma 2.17 (iii) to this sequence and noting $c_\varnothing = (\bigwedge_{v\in S} f_v)(c_S)$. To finish the proof, we show that any set S chosen as above has the property stated in (i). To do this, we fix such an S and a Stark system $(c_T)_T$ with $c_S = 0$, and need to show, for every $T\in\mathcal{P}$, that c_T is annihilated by $\mathrm{Fitt}_R^{r+|S|}(M_S^*)$. Now, after replacing T by $T\cup S$, we can assume T contains S, and hence that $\Omega = (\ker\rho_{\varnothing,S}) = \ker(\rho_{\varnothing,T})$. From the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega & \longrightarrow & Z_\varnothing & \xrightarrow{\rho_{\varnothing,S}} & Z_S & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow{\scriptstyle \rho_{S,T}} & & \\
0 & \longrightarrow & \Omega & \longrightarrow & Z_\varnothing & \xrightarrow{\rho_{\varnothing,T}} & Z_T & \longrightarrow & 0
\end{array}$$

we thus deduce $\rho_{S,T}$ is bijective. In this case, therefore, (P$_5$) gives a split-exact sequence

$$0 \longrightarrow M_S \longrightarrow M_T \xrightarrow{\oplus_{v\in T\setminus S} f_v} R^{|T\setminus S|} \longrightarrow 0,$$

and hence an isomorphism $M_T^* \cong M_S^* \oplus R^{|T\setminus S|}$. This in turn induces an isomorphism

$$\bigwedge\nolimits_R^{r+|T|} M_T^* \cong \bigoplus\nolimits_{i=0}^{i=|T\setminus S|} (\bigwedge\nolimits_R^{r+|T|-i} M_S^*) \otimes_R (\bigwedge\nolimits_R^{i} R^{|T\setminus S|})$$

which, upon dualising, gives an exact sequence

$$0 \longrightarrow \bigoplus\nolimits_{i=r+|S|+1}^{r+|T|} \bigcap\nolimits_R^i M_S \longrightarrow \bigcap\nolimits_R^{r+|T|} M_T \xrightarrow{\wedge_{v\in T\setminus S} f_v} \bigcap\nolimits_R^{r+|S|} M_S.$$

In particular, since $c_T \in \ker(\wedge_{v\in T\setminus S} f_v)$, one has $c_T \in \bigoplus_{i=r+|S|+1}^{r+|T|} \bigcap_R^i M_S$. Thus, it suffices to note that, by Lemma 2.11 (v), the ideal $\mathrm{Fitt}_R^{r+|S|}(M_S^*)$ annihilates $\bigwedge_R^i M_S^*$ for each $i \in \{r+|S|+1,\dots,r+|T|\}$, and hence that $\mathrm{Fitt}_R^{r+|S|}(M_S^*)$ annihilates $\bigoplus_{i=r+|S|+1}^{r+|T|} \bigcap_R^i M_S$. □

(A.7) Example. Let K/k be a finite abelian group with Galois group $G \coloneqq \mathrm{Gal}(K/k)$. Let (S,V,T) be a Rubin datum for K/k, and suppose that the Rubin–Stark Conjecture holds for all data $(S\cup S', V\cup S', T)$ as S' ranges over $\mathcal{P}(\mathcal{Q})$ for a set $\mathcal{Q}$ of finite places of k that is disjoint from T and comprises only places that split completely in K/k. Then the collection $(\varepsilon^{V\cup S'}_{K/k,S\cup S',T})_{S'\in\mathcal{P}}$ of Rubin–Stark elements is a Stark system in the above sense, with $R = \mathbb{Z}[G]$, $M_S = \mathcal{O}^\times_{K,S,T}$ and Z_S equal to the (S_K,T_K)-ray class group $\mathrm{Cl}_{K,S,T}$ of K. Hence, in this case, Theorem A.6 (ii) implies $\mathrm{im}(\varepsilon^V_{K/k,S,T})$ is contained in the $\mathbb{Z}[G]$-characteristic ideal of the subgroup of $\mathrm{Cl}_{K,S,T}$ generated by the classes of places w in $\mathcal{Q}_K$. (See also [15, Th. 1.1] for an application of this observation.)

A.3. The Eagon–Northcott complex

In this section we review a certain family of complexes introduced by Eagon and Northcott in [33] (see also [36, App. 2, § A2H]). These complexes generalise the classical Koszul complex (see [35, § A2.6] or [84, App. C] for unifying frameworks) and computes Fitting ideals. To explain its construction, we fix a morphism $\phi\colon F_0 \to F_1$ of free R-modules F_0 and F_1 of finite ranks d and $d-r$, respectively. For $i \in \{0,\dots,r-1\}$ we write $\mathrm{Sym}_{r-i} F_1$ for the $(r-i)$-th symmetric power of F_1. Then the dual of the multiplication map $\mu_i\colon F_1 \otimes_R (\mathrm{Sym}_{r-i-1} F_1) \to \mathrm{Sym}_{r-i} F_1$ is a map $\mu_i^*\colon (\mathrm{Sym}_{r-i} F_1)^* \to F_1^* \otimes_R (\mathrm{Sym}_{r-i-1} F_1)^*$ and we use this to define a composite

$$\begin{aligned}\partial_i\colon (\mathrm{Sym}_{r-i} F_1)^* \otimes_R \bigwedge\nolimits_R^{d-i} F_0 &\xrightarrow{\mu_i^*\otimes\phi^{(d-i)}} \big(F_1^* \otimes_R (\mathrm{Sym}_{r-i-1} F_1)^*\big) \otimes_R \big(F_1 \otimes_R \bigwedge\nolimits_R^{d-i-1} F_0\big)\\ &\longrightarrow (\mathrm{Sym}_{r-i-1} F_1)^* \otimes_R \bigwedge\nolimits_R^{d-i-1} F_0.\end{aligned}$$

Here $\phi^{(d-i)}\colon \bigwedge_R^{d-i} F_0 \to F_1 \otimes_R \bigwedge_R^{d-i-1} F_0$ is defined by means of

$$\phi^{(d-i)}(m_1 \wedge \dots \wedge m_{d-i}) = \sum\nolimits_{j\in[d-i]} (-1)^{j+1} \phi(m_j) \otimes m_1 \wedge \dots \wedge \widehat{m_j} \wedge \dots \wedge m_{d-i},$$

where we write $\widehat{m_j}$ to mean omission of theelement m_j, and the second arrow is induced by the canonical evaluation map $F_1^* \otimes_R F_1 \to R$.
It is proved in [35, Th. A2.10 (a)] that the sequence

$$\begin{aligned}(\mathrm{Sym}_r F_1)^* \otimes_R \bigwedge\nolimits_R^d F_0 &\xrightarrow{\partial_0} (\mathrm{Sym}_{r-i-1} F_1)^* \otimes_R \bigwedge\nolimits_R^{d-i-1} F_0 \xrightarrow{\partial_1} \dots\\ \dots \longrightarrow F_1^* \otimes_R \bigwedge\nolimits_R^{d-r+1} F_0 &\xrightarrow{\partial_{r-1}} \bigwedge\nolimits_R^{d-r} F_0 \xrightarrow{\bigwedge_R^{d-r}\phi} \bigwedge\nolimits_R^{d-r} F_1\end{aligned}$$

constitutes a complex $C^\bullet_{\mathrm{EN}}(\phi)$ of (finitely generated, free) R-modules in which the final term $\bigwedge_R^{d-r} F_1$ is considered (by convention) as placed in degree zero.
The following well-known property of $C^\bullet_{\mathrm{EN}}(\phi)$ will play a crucial role in later arguments.

(A.8) Proposition. *For every morphism $\phi\colon F_0 \to F_1$ of free R-modules as above, one has that $\mathrm{Fitt}_R^0(\mathrm{coker}\,\phi)$ annihilates $H^i(C^\bullet_{\mathrm{EN}}(\phi))$ for every $i \in \mathbb{Z}$.*

Proof. This result is stated without proof in [36, Th. A2.59]. For completeness, we therefore present a proof. For this, we follow an argument used by Buchsbaum and Rim [9] to prove the analogous property for the Buchsbaum–Rim complex.

Let $S \coloneqq R[X_{ij} \mid i \in [d-r], j \in [d]]$ be the polynomial ring in $d(d-r)$ commuting variables and consider the map $\Phi\colon S^{\oplus d} \to S^{\oplus(d-r)}$ that is represented by the matrix $(X_{ij})_{ij}$. It then is a result of Northcott [83] that $\mathrm{Fitt}^0_S(\operatorname{coker}\Phi)$ is an ideal of grade $d-(d-r)+1 = r+1$. It therefore follows from [35, Th. A2.10 (c)] that $C^\bullet_{\mathrm{EN}}(\Phi)$ is acylic. (This is what is referred to as the 'generic case' in loc. cit.). In particular, $C^\bullet_{\mathrm{EN}}(\Phi)$ furnishes a projective resolution of

$$H^0(C^\bullet_{\mathrm{EN}}(\Phi)) = \operatorname{coker}\big\{ \bigwedge\nolimits_S^{d-r} S^{\oplus d} \xrightarrow{\bigwedge_S^{d-r}\Phi} \bigwedge\nolimits_S^{d-r} S^{\oplus(d-r)}\big\} \cong S/\operatorname{Fitt}^0_S(\operatorname{coker}\Phi).$$

Let ϕ be represented by a matrix $A = (a_{ij})_{ij}$ and define $\overline{S}$ to be the quotient of S by the ideal generated by $X_{ij} - a_{ij}$ for $1 \leq i \leq d-r$ and $1 \leq j \leq d$. Since Eagon–Northcott complexes behave well under base change, the isomorphism $\overline{S} \cong R$ that sends the class of X_{ij} to a_{ij} induces an isomorphism $C^\bullet_{\mathrm{EN}}(\phi) \cong C^\bullet_{\mathrm{EN}}(\Phi) \otimes^{\mathbb{L}}_S \overline{S}$. It follows that for every $i \geq 0$ one has

$$H^i(C^\bullet_{\mathrm{EN}}(\phi)) = \operatorname{Tor}_i^S(S/\operatorname{Fitt}^0_S(\operatorname{coker}\Phi), \overline{S}).$$

This description shows that $H^i(C^\bullet_{\mathrm{EN}}(\phi))$ is annihilated by the image of $\mathrm{Fitt}^0_S(\operatorname{coker}\Phi)$ under $\overline{S} \cong R$. By Lemma 2.11 (iv), this image is equal to $\mathrm{Fitt}^0_R(\overline{S} \otimes_S \operatorname{coker}\Phi) = \mathrm{Fitt}^0_R(\operatorname{coker}\phi)$, whence the claimed annihilation result follows. □

A.4. Determinants and biduals

We next prove a useful technical result that relates the determinant of a perfect complex to an appropriate bidual of its cohomology.

(A.9) Proposition. *Let R be a G_2-ring, d a natural number and r a non-negative integer with $r \leq d$. Let $C^\bullet$ be a complex of R-modules $F_0 \xrightarrow{\phi} F_1$, in which the first term is a free R-module of rank d that is placed in degree zero and F_1 is a free R-module of rank $d-r$ that is placed in degree one. Then the following claims are valid.*

(i) The image of the canonical map

$$\vartheta_\phi\colon \operatorname{Det}_R(C^\bullet) \coloneqq \big(\bigwedge\nolimits_R^d F_0\big) \otimes_R \big(\bigwedge\nolimits_R^{d-r} F_1^*\big) \to \bigwedge\nolimits_R^r F_0,$$
$$a \otimes (\wedge_{i\in[d-r]} f_i) \mapsto (-1)^{r(d-r)} \cdot \big(\wedge_{i\in[d-r]}(f_i \circ \phi)\big)(a)$$

is contained in $\bigcap_R^r H^0(C^\bullet)$, and there is also an inclusion of ideals of R

$$\mathrm{Fitt}^0_R(H^1(C^\bullet)) \cdot \mathrm{Ann}_R(\mathrm{Ext}^1_R(R/\mathrm{Fitt}^0_R(H^1(C^\bullet)), R)) \subseteq \mathrm{Ann}_R\big(\bigcap\nolimits_R^r H^0(C^\bullet)/(\operatorname{im}\vartheta_\phi)\big).$$

(ii) We have an inclusion of ideals of R

$$\mathrm{Fitt}^0_R(H^1(C^\bullet)) \subseteq \big\{ f(a) \mid a \in \operatorname{im}(\vartheta_\phi), f \in \bigwedge\nolimits_R^r H^0(C^\bullet)^* \big\}$$

with pseudo-null cokernel.

(iii) There exists a canonical injective map

$$\widetilde{\vartheta_\phi}\colon \mathrm{Fitt}^0_R(H^1(C^\bullet))^* \otimes_R \operatorname{Det}_R(C^\bullet) \to \bigcap\nolimits_R^r H^0(C^\bullet),$$

that sends $\mathrm{id}_R \otimes a$ to ϑ_ϕ. The cokernel of $\widetilde{\vartheta_\phi}$ is R-torsion-free and annihilated by $\mathrm{Fitt}^0_R(H^1(C^\bullet))$. In particular, $\widetilde{\vartheta_\phi}$ is bijective if $H^1(C^\bullet)$ is R-torsion.

(iv) Suppose $D^\bullet$ is a perfect complex of R-modules that fits into an exact triangle of the form

$$C^\bullet \longrightarrow D^\bullet \longrightarrow R^n[0] \longrightarrow \quad . \tag{A.10}$$

Then $D^\bullet$ admits a representative $R^{d+n} \xrightarrow{\psi} R^{d-r}$, in which the first term is placed in degree zero, and there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Det}_R(D^\bullet) & \xrightarrow{\vartheta_\psi} & \bigcap_R^{r+n} H^0(D^\bullet) \\ \downarrow\simeq & & \downarrow (-1)^{rn}\wedge_{i\in[n]} f_i \\ \operatorname{Det}_R(C^\bullet) & \xrightarrow{\vartheta_\phi} & \bigcap_R^r H^0(C^\bullet). \end{array}$$

Here the first vertical map is the isomorphism induced by (A.10) and the canonical identification $\mathrm{Det}_R(R^n) \cong (R, n)$*, and the second vertical map is induced, via Lemma 2.17 (b), by the map* $(f_i)_{i\in[n]}\colon H^0(D^\bullet) \to R^n$ *in the long exact cohomology sequence of (A.10).*

Proof. The first part of (i) is proved by the argument of Lemma 2.37. As for the second part of (i), it is convenient to first prove (ii) and (iii) and then complete the proof of (i).
The proof of [27, Prop. A.2 (ii)] shows that

$$\Big\{f(a) \mid a \in \mathrm{im}(\vartheta_\phi),\ f \in {\textstyle\bigwedge_R^r} F_0^*\Big\} = \mathrm{Fitt}_R^0(H^1(C^\bullet)), \tag{A.11}$$

and so (ii) follows from the same argument as used in Lemma A.3 (b).
To prove (iii), we will first show that $\mathrm{im}(\vartheta_\phi) \subseteq \mathrm{Fitt}_R^0(H^1(C^\bullet)) \cdot \bigwedge_R^r F_0$ (and hence the rule used to define the map in claim (c) is well-defined). Choose an ordered basis $\{b_i\}_{i\in[d]}$ of F_0 and consider the basis of $\bigwedge_R^r F_0$ given by the family $\{\bigwedge_{i\in I} b_i\}_I$, where I runs through all (ordered) subsets of $[d]$ of cardinality r. Any element x of $\mathrm{im}(\vartheta_\phi)$ is then equal to $\sum_I (\bigwedge_{i\in I} b_i^*)(x)\cdot(\bigwedge_{i\in I} b_i)$ and so, by (A.11), belongs to $\mathrm{Fitt}_R^0(H^1(C^\bullet)) \cdot \bigwedge_R^r F_0$, as claimed.
To proceed, we recall the image of the map $\bigwedge_R^{d-r}\phi\colon \bigwedge_R^{d-r} F_0 \to \bigwedge_R^{d-r} F_1$ is, by definition, $\mathrm{Fitt}_R^0(H^1(C^\bullet)) \cdot \bigwedge_R^{d-r} F_1$. Hence, writing A for the kernel of this map, and B for the cokernel of the map $\partial_{r-1}\colon F_1^* \otimes_R \bigwedge_R^{d-r+1} F_0 \to \bigwedge_R^{d-r} F_0$, we obtain an exact commutative diagram

$$\begin{array}{ccccc}
F_1^* \otimes_R \bigwedge_R^{d-r+1} F_0 & \xrightarrow{\partial_{r-1}} & \bigwedge_R^{d-r} F_0 & \longrightarrow\!\!\!\!\rightarrow & B \\
\downarrow & & \| & & \downarrow \\
A & \hookrightarrow & \bigwedge_R^{d-r} F_0 & \longrightarrow\!\!\!\!\rightarrow & \mathrm{Fitt}_R^0(H^1(C^\bullet)) \otimes_R \bigwedge_R^{d-r} F_1 \\
\downarrow\!\!\downarrow & & & & \\
H^{-1}(C^\bullet_{\mathrm{EN}}(\phi)), & & & &
\end{array} \tag{A.12}$$

where $C^\bullet_{\mathrm{EN}}(\phi)$ denotes the Eagon–Northcott complex that was introduced in § A.3. By applying the Snake Lemma to this diagram, one then obtains an exact sequence

$$0 \longrightarrow H^{-1}(C^\bullet_{\mathrm{EN}}(\phi)) \longrightarrow B \longrightarrow \mathrm{Fitt}_R^0(H^1(C^\bullet)) \otimes_R \bigwedge_R^{d-r} F_1 \longrightarrow 0. \tag{A.13}$$

Now, by dualising the top row of (A.12) we obtain the top row in the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & B^* \otimes_R \bigwedge_R^d F_0 & \longrightarrow & \big(\bigwedge_R^{d-r} F_0\big)^* \otimes_R \bigwedge_R^d F_0 & \longrightarrow & F_1 \otimes_R \big(\bigwedge_R^{d-r+1} F_0\big)^* \otimes_R \bigwedge_R^d F_0 \\
 & & \downarrow & & \downarrow\simeq & & \downarrow\simeq \\
0 & \longrightarrow & \bigcap_R^r H^0(C) & \longrightarrow & \bigwedge_R^r F_0 & \longrightarrow & F_1 \otimes_R \bigwedge_R^{r-1} F_0,
\end{array}$$

where the bottom row is exact by Lemma 2.17 (a) and the vertical isomorphisms are instances of the isomorphism

$$\big(\bigwedge_R^{d-i} F_0\big)^* \otimes_R \bigwedge_R^d F_0 \xrightarrow{\simeq} \bigwedge_R^i F_0, \quad \theta \otimes x \mapsto \theta(x) \tag{A.14}$$

that exists for every $i \in \{0, \dots, d\}$. This shows that $B^* \otimes_R \bigwedge_R^d F_0$ identifies with $\bigcap_R^r H^0(C^\bullet)$. Tensoring the dual of (A.13) with $\bigwedge_R^d F_0$ therefore produces a canonical exact sequence

$$\mathrm{Fitt}_R^0(H^1(C^\bullet))^* \otimes_R \big(\bigwedge_R^{d-r} F_1\big)^* \otimes_R \big(\bigwedge_R^d F_0\big) \hookrightarrow \bigcap_R^r H^0(C^\bullet) \to H^{-1}(C^\bullet_{\mathrm{EN}}(\phi))^* \otimes_R \big(\bigwedge_R^d F_0\big).$$

In connection with this exact sequence we note that the first map is defined as the composite of the dual map of $\bigwedge_R^{d-r}\phi$ and the isomorphism (A.14) for $i = r$. If we define $\widetilde{\vartheta_\phi}$ to be this map, then an explicit check therefore shows that $\widetilde{\vartheta_\phi}(\mathrm{id}_R \otimes a) = \vartheta_\phi(a)$, as claimed in (iii). The above exact sequence then shows that the cokernel of $\widetilde{\vartheta_\phi}$ is isomorphic to a submodule of $H^{-1}(C^\bullet_{\mathrm{EN}}(\phi))^* \otimes_R \big(\bigwedge_R^d F_0\big)$. As such, this cokernels must be R-torsion free.

To further analyse $H^{-1}(C^\bullet_{\mathrm{EN}}(\phi))^* \otimes_R \big(\bigwedge_R^d F_0\big)$, and thereby justify the rest of (iii), we note Proposition A.8 implies $\mathrm{Fitt}^0_R(\operatorname{coker}\phi)$ annihilates the cohomology of $C^\bullet_{\mathrm{EN}}(\phi)$ in every degree. In particular, the module $H^{-1}(C^\bullet_{\mathrm{EN}}(\phi))$ is annihilated by $\mathrm{Fitt}^0_R(H^1(C^\bullet))$. If $H^1(C^\bullet)$ is R-torsion, then its Fitting ideal contains a nonzero divisor and so $H^{-1}(C^\bullet_{\mathrm{EN}}(\phi))$ must be R-torsion.

Returning now to the final claim of (i), we observe that a slight adaptation of (A.13) leads to the exact sequence

$$0 \longrightarrow B/H^{-1}(C^\bullet_{\mathrm{EN}}(\phi)) \longrightarrow \bigwedge\nolimits_R^{d-r} F_1 \longrightarrow (R/\operatorname{Fitt}^0_R(H^1(C^\bullet))) \otimes_R \bigwedge\nolimits_R^{d-r} F_1 \longrightarrow 0.$$

Dualising and tensoring with $\bigwedge_R^r F_0$ then similarly leads to a diagram of the form

$$\begin{array}{ccccc}
\big(\bigwedge_R^{d-r} F_1\big)^* \otimes_R \big(\bigwedge_R^{d} F_0\big) & \longrightarrow & \big(B/H^{-1}(C^\bullet_{\mathrm{EN}}(\phi))\big)^* & \longrightarrow & \mathrm{Ext}^1_R((R/\operatorname{Fitt}^0_R(H^1(C^\bullet))), R) \\
\Vert & & \big\downarrow & & \\
\big(\bigwedge_R^{d-r} F_1\big)^* \otimes_R \big(\bigwedge_R^{d} F_0\big) & \xrightarrow{\ \vartheta_\phi\ } & \bigcap_R^r H^0(C^\bullet) & & \\
 & & \big\downarrow & & \\
 & & H^{-1}(C^\bullet_{\mathrm{EN}}(\phi))^* & &
\end{array}$$

in which all rows and columns are exact. Given this, the annihilation statement claimed in (i) now follows from Proposition A.8.

To prove (iv), we write $h = h^\bullet \colon R^n[0] \to C^\bullet[-1]$ for the morphism in $D(R)$ that appears in the triangle (A.10). One then has $D^\bullet = \operatorname{cone}(h)[1]$, and so $D^\bullet$ is represented by $[P_0 \xrightarrow{\psi} P_1]$ with $P_0 \coloneqq R^n \oplus F_0$, $P_1 \coloneqq F_1$, and $\psi \colon P_0 \to P_1$ the map defined by sending $a \oplus b \in R^n \oplus F_0$ to $h^1(a) + \phi(b)$. This verifies the first part of claim (d).

To proceed, we write $\{e_i : i \in [n]\}$ for the standard basis of R^n (regarded as elements of P_0), and $e_i^* \colon R^n \to R$ for the dual of e_i for $i \in [n]$. The isomorphism $\mathrm{Det}_R(D^\bullet) \cong \mathrm{Det}_R(C^\bullet)$ induced by the triangle (A.10) can then be explicitly described as

$$\begin{aligned}
\mathrm{Det}_R(D^\bullet) = \big(\bigwedge\nolimits_R^{d+n}(R^n \oplus F_0)\big) \otimes_R \big(\bigwedge\nolimits_R^{d-r} F_0^*\big) \to \big(\bigwedge\nolimits_R^{d} F_0\big) \otimes_R \big(\bigwedge\nolimits_R^{d-r} F_0^*\big) = \mathrm{Det}_R(C^\bullet) \\
h \otimes g \mapsto (-1)^{n\cdot d} \cdot (\wedge_{i\in[n]} e_i^*)(h) \otimes g.
\end{aligned}$$

In addition, we may calculate that for $g = \wedge_{i\in[r]} g_i$ one has

$$\begin{aligned}
(-1)^{nd} \cdot \vartheta_\phi\big((\wedge_{i\in[n]} e_i^*)(a) \otimes g\big) &= (-1)^{nd} \cdot \big((\wedge_{i\in[d-r]}(g_i \circ \phi)) \circ (\wedge_{i\in[n]}(e_i^* \circ h^1))\big)(a) \\
&= (-1)^{nd} \cdot \big((\wedge_{i\in[n]} e_i^*) \wedge (\wedge_{i\in[d-r]}(g_i \circ \phi))\big)(a) \\
&= (-1)^{nd+n(d-r)} \cdot \big((\wedge_{i\in[d-r]}(g_i \circ \phi)) \wedge (\wedge_{i\in[n]}(e_i^* \circ h^1))\big)(a) \\
&= (-1)^{nr} \cdot \big((\wedge_{i\in[n]} e_i^*) \circ (\wedge_{i\in[d-r]}(g_i \circ \phi))\big)(a) \\
&= (-1)^{r(d+n-r)} \cdot \big((\wedge_{i\in[n]} f_i) \circ \vartheta_\psi\big)(a).
\end{aligned}$$

Here we have used for the last equality that $\vartheta_\psi(a)$ belongs to $\bigcap_R^{r+n} H^0(D^\bullet)$ and so its image under $\wedge_{i\in[n]} e_i^*$ is equal to its image under $\wedge_{i\in[n]} f_i$. (Recall that each e_i induces the map $f_i \colon H^0(D^\bullet) \to R^n$ on cohomology). This concludes the proof of the Proposition. □

A.5. Algebraic Stark systems

Let $\mathcal{Q}$ be a set, and regard its power set $\mathcal{P} \coloneqq \mathcal{P}(\mathcal{Q})$ as a partially ordered set with respect to inclusion. Let R be a G_2-ring and suppose to be given a family $(C_S^\bullet)_{S\in\mathcal{P}}$ of perfect complexes of R-modules that satisfies the following hypotheses.

(A.15) Hypothesis. Assume the following conditions are valid.

(a) $C^\bullet_\varnothing$ admits a representative of the form $[F_0 \to F_1]$ with F_0 and F_1 finitely generated free R-modules. (Here the first term is placed in degree zero.)

(b) The integer $r := \mathrm{rk}_R(F_0) - \mathrm{rk}_R(F_1)$ is non-negative.

(c) For every pair of sets $S, S' \in \mathcal{P}$ with $S \subseteq S'$ there exists a specified exact triangle

$$C^\bullet_S \xrightarrow{\iota_{S,S'}} C^\bullet_{S'} \xrightarrow{\oplus_{v\in S'\setminus S} f_{S',v}} \bigoplus\nolimits_{v\in S'\setminus S} R[0] \longrightarrow \tag{A.16}$$

such that $f_{S',v} \circ \iota_{S,S'} = f_{S,v}$. We will therefore simply write f_v instead of $f_{S',v}$.

(A.17) Remark.

(i) If $(C^\bullet_S)_{S\in\mathcal{P}}$ satisfies Hypothesis A.15, then we obtain a system $(M_S, \iota'_{S,S'})_{S\in\mathcal{P}}$ as in § A.2 with $M_S := H^0(C^\bullet_S)$ and $\iota'_{S,S'}$ the map $H^0(\iota_{S,S'}) : H^0(C^\bullet_S) \to H^0(C^\bullet_{S'})$. In particular, we obtain a module $\mathrm{StS}^r(\{C^\bullet_S, \iota_{S,S'}\}_{S\in\mathcal{P}})$ of Stark systems of rank r associated to $(C^\bullet_S)_{S\in\mathcal{P}}$.

(ii) Hypothesis A.15 and Proposition A.9 (d) combine to imply that $C^\bullet_S$ has a representative $[P_0 \overset{\phi_S}{\to} P_1]$ in which P_0 and P_1 are free R-modules of ranks $\mathrm{rk}_R(F_0) + |S|$ and $\mathrm{rk}_R(F_1)$.

We can now state our main result on Stark systems in this setting.

(A.18) Theorem. *Let R be a G_2-ring and $(C^\bullet_S)_{S\in\mathcal{P}}$ a family of perfect complexes of R-modules satisfying Hypothesis A.15. Write Ω for the image of the map $\sum g_v \colon \bigoplus_{v\in\mathcal{Q}} R \to H^1(C^\bullet_\varnothing)$, where each $g_v \colon R \to H^1(C^\bullet_\varnothing)$ is the boundary map in the long exact sequence in cohomology associated with (A.16) for $S' = \{v\}$ and $S = \varnothing$. Then the following claims are valid.*

(i) There exist $S \in \mathcal{P}$ with $\Omega = \mathrm{im}(\sum_{v\in S} g_v)$, and for such S, the ideal $\mathrm{Fitt}^0_R(H^1(C^\bullet_S)) = \mathrm{Fitt}^0_R(H^1(C^\bullet_\varnothing)/\Omega)$ annihilates the kernel of the 'projection map'

$$\mathrm{pr}_S \colon \mathrm{StS}^r(\{C^\bullet_S, \iota_{S,S'}\}_{S\in\mathcal{P}}) \to \bigcap\nolimits_R^{r+|S|} H^0(C^\bullet_S).$$

(ii) There exists a well-defined homomorphism of R-modules

$$F \colon \varprojlim\nolimits_{S\in\mathcal{P}} \mathrm{Det}_R(C^\bullet_S) \to \mathrm{StS}^r(\{C^\bullet_S, \iota_{S,S'}\}_{S\in\mathcal{P}}), \quad (a_S)_{S\in\mathcal{P}} \mapsto (\vartheta_{\phi_S}(a_S))_{S\in\mathcal{P}},$$

where the transition maps on the left are the isomorphisms $\mathrm{Det}_R(C^\bullet_{S'}) \overset{\simeq}{\to} \mathrm{Det}_R(C^\bullet_S)$ induced by (A.16). The kernel of this map is annihilated by $\mathrm{Ann}_R(\mathrm{Fitt}^0_R(H^1(C^\bullet_\varnothing)/\Omega)^*)$, and its cokernel is annihilated by $\mathrm{Fitt}^0_R(H^1(C^\bullet_\varnothing)/\Omega)^2 \cdot \mathrm{Ann}_R(\mathrm{Ext}^1_R(R/\mathrm{Fitt}^0_R(H^1(C^\bullet_\varnothing)/\Omega), R))$.

(iii) For every $c \in \mathrm{StS}^r(\{C^\bullet_S, \iota_{S,S'}\}_{S\in\mathcal{P}})$, one has $\mathrm{im}(c_\varnothing) \subseteq \mathrm{char}_R(\Omega)$, $\mathrm{Fitt}^0_R(H^1(C^\bullet_\varnothing)/\Omega) \cdot c_\varnothing \subseteq \mathrm{im}(\vartheta_{\phi_\varnothing})$ and $\mathrm{Fitt}^0_R(H^1(C^\bullet_\varnothing)/\Omega) \cdot \mathrm{im}(c_\varnothing) \subseteq \mathrm{Fitt}^0_R(H^1(C^\bullet_\varnothing))^{**}$.

Proof. Theorem A.6 (i) implies both the existence of $S \in \mathcal{P}$ with $\Omega = \mathrm{im}(\sum_{v\in S} g_v)$ and that $\iota(\mathrm{Fitt}^{r+|S|}_R(H^0(C^\bullet)^*))$ annihilates $\ker(\mathrm{pr}_S)$. To finish the proof of (i), it is thus enough to note that by Lemma 2.36 one has $\mathrm{Fitt}^0_R(H^1(C^\bullet_S)) \subseteq \iota(\mathrm{Fitt}^{r+|S|}_R(H^0(C^\bullet)^*))$.
The first part of (ii) follows from Proposition A.9 (iv), and the second part (regarding the kernel of F) follows from Proposition A.9 (iii). To prove the second part we fix a set S with $\Omega = \ker(H^1(C^\bullet_\varnothing) \to H^1(C^\bullet_S))$, and hence $H^1(C^\bullet_S) \cong H^1(C^\bullet_\varnothing)/\Omega$ (and note that any S' that contains S has the same property). We also fix $c = (c_X)_{X\in\mathcal{P}} \in \mathrm{StS}^r(\{C^\bullet_S, \iota_{S,S'}\}_{S\in\mathcal{P}})$ and $x \in \mathrm{Fitt}^0(H^1(C^\bullet_S)) \cdot \mathrm{Ann}_R(\mathrm{Ext}^1_R(R/\mathrm{Fitt}^0_R(H^1(C^\bullet_S)), R))$. Then Proposition A.9 (i) implies there exists $z_S \in \mathrm{Det}_R(C^\bullet_S)$ with $\vartheta_{\phi_S}(z_S) = xc_S$. Since all transition maps in the inverse limit $\varprojlim_{S\in\mathcal{P}} \mathrm{Det}_R(C^\bullet_S)$ are isomorphisms, we can then lift z_S to an element z of said inverse limit. Now, $F(z)$ is a Stark system with $\vartheta_{\phi_S}(z_S) = xz_S$, and so hence $F(z) - xc \in \ker(\mathrm{pr}_S)$. By claim (a) we therefore have $yxz = yF(z)$ for any $y \in \mathrm{Fitt}^0_R(H^1(C^\bullet_S))$. This implies that $\mathrm{cok}(F)$ is annihilated by $\mathrm{Fitt}^0_R(H^1(C^\bullet_S))^2 \cdot \mathrm{Ann}_R(\mathrm{Ext}^1_R(R/\mathrm{Fitt}^0_R(H^1(C^\bullet_S)), R))$, as claimed in the second part of (ii).
Then the first part of (iii) is a special case of Theorem A.6 (ii), the second part is a consequence of (ii), and and the third part follows from the second by Proposition A.9 (ii). □

Part II. Arithmetic applications

In the remainder of the article we apply the theory developed above to improve a range of existing results concerning special value conjectures.

8. Relaxed Nekovář structures and Kato's conjecture

In this section we explain how to apply Theorem 4.20 for suitably modified relaxed Nekovář structures in order to study the 'generalised Iwasawa main conjecture' formulated by Kato in [57, 58]. For brevity, in the sequel we shall refer to the latter conjecture as 'Kato's conjecture'.

8.1. The general strategy

We first review Kato's conjecture. To do this, we fix a smooth projective variety X over k for which the motive $M = h^i(X)(j)$ has coefficients in a semi-simple $\mathbb{Q}$-algebra A. Then, for each prime ℓ, the ℓ-adic realisation $V_\ell(M) \coloneqq H^i_{\text{ét}}(X_{k^c}, \mathbb{Q}_\ell)(j)$ is a finitely generated module over the semisimple $\mathbb{Q}_\ell$-algebra $A_\ell \coloneqq A \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ that is endowed with a continuous commuting action of G_k. It is conjectured that, for every finite prime $\mathfrak{q} \nmid \ell$ of k, the characteristic polynomial

$$\mathrm{Eul}_{\mathfrak{q}}(M, x) \coloneqq \det_{A_\ell}(1 - \mathrm{Frob}_{\mathfrak{q}}^{-1} x \mid V_\ell(M)^*(1)^{I_{\mathfrak{q}}}) \in A_\ell[x]$$

belongs to $A[x]$ and is independent of ℓ. Assuming this, for each finite set of places S of k that contains both S_k^∞ and all places at which M has bad reduction, the S-truncated motivic L-series of M is defined via the $(A \otimes_{\mathbb{Q}} \mathbb{C})$-valued infinite product

$$L_S(M, s) \coloneqq \prod\nolimits_{\mathfrak{q} \notin S} \mathrm{Eul}_{\mathfrak{q}}(M, \mathrm{Frob}_{\mathfrak{q}}^{-1} \cdot \mathrm{N}\mathfrak{q}^{-s})^{-1}.$$

This product converges if the real part of s is large-enough and we assume (as is conjectured) it has a meromorphic continuation to $s = 0$ and write $L_S^*(M, 0)$ for its leading term at $s = 0$. It can be shown that $L_S^*(M, 0)$ belongs to $(\mathbb{R} \otimes_{\mathbb{Q}} A)^\times$ and to study this element one fixes a prime number p together with a Gorenstein $\mathbb{Z}_p$-order $\mathcal{A}_p$ in A_p for which there exists a (full) G_k-stable sublattice $T \coloneqq T_p(M)$ of $V_p(M)$ for which T, and hence also $\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1))$, is free as an $\mathcal{A}_p$-module. We assume $S_k^p \subseteq S$ and use the complex

$$C_S(T) \coloneqq C(\mathscr{F}_{\mathrm{rel}}(T, S)) = \mathrm{RHom}_{\mathcal{A}_p}(\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{O}_{k,S}, T), \mathcal{A}_p)[-2]$$

constructed in Proposition 3.45. Here $\mathscr{F}_{\mathrm{rel}}(T, S)$ is the relaxed Nekovář structure specified in Example 3.15 (i) so that the second equality follows from Examples 3.33 and 3.15(iii).

Then, modulo standard conjectures, the theory developed by Bloch and Kato [5], as interpreted and extended by Kato [57, 58] and Fontaine–Perrin-Riou [41], gives rise to a canonical rank-one A-module $\Xi(M)$, called the 'fundamental line' of M, as well as canonical 'period-regulator' isomorphisms of invertible $\mathbb{R} \otimes_{\mathbb{Q}} A$-modules, resp. A_p-modules

$$\lambda_M \colon \mathbb{R} \otimes_{\mathbb{Q}} A \xrightarrow{\simeq} \mathbb{R} \otimes_{\mathbb{Q}} \Xi(M), \quad \text{resp. } \vartheta_{M,S} \colon \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Det}_{\mathcal{A}_p}(C_S(T)) \xrightarrow{\simeq} \mathbb{Q}_p \otimes_{\mathbb{Q}} \Xi(M).$$

Writing $M^*(1)$ for the Kummer dual of M, we can now recall Kato's conjecture for M relative to $\mathcal{A}_p$. (This conjecture was first formulated in [57] and is shown in [19, § 2] to be equivalent, in the subsequent terminology of [18], to the 'equivariant Tamagawa number conjecture' for the pair $(M, \mathcal{A}_p)$.)

(8.1) Conjecture (TNC$(M, \mathcal{A}_p)$). *One has*

$$\lambda_M(L_S^*(M^*(1), 0)) \in \Xi(M) \quad \textit{and} \quad \mathcal{A}_p \cdot \lambda_M(L_S^*(M^*(1), 0)) = \vartheta_{M,S}\big(\mathrm{Det}_{\mathcal{A}_p}(C_S(T))\big).$$

For each Galois extension F of k we now consider the free $\mathcal{A}_p[\mathcal{G}_F]$-module

$$T_{F/k} \coloneqq \mathrm{Ind}_{G_k}^{G_F}(T) = T \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{G}_F]$$

with G_k-action given by $\sigma\cdot(a\otimes b) := (\sigma a)\otimes(x\overline{\sigma}^{-1})$ for $\overline{\sigma}\in\mathcal{G}_F$ the restriction of $\sigma\in G_k$ to F. For each place $v\in\Pi_k^\infty$ we also fix an extension of v to k^{c} and write G_v for the corresponding decomposition subgroup of G_k. Given an abelian extension K of k, we denote the restriction of τ_v to K by $\tau_{K,v}$ and define an $\mathcal{A}_p[\langle\tau_{K,v}\rangle]$-module by setting

$$\mathcal{Y}_K(T,v) := \begin{cases} T(-1)^{\tau_v=1} & \text{if } \tau_{K,v}=\mathrm{id}_K,\\ T(-1) & \text{if } \tau_{K,v}\neq\mathrm{id}_K,\end{cases}$$

where $M^{\tau_v=\pm1}$ denotes the maximal submodule of a G_k-module M on which τ_v acts as ±1. We may then define a subset of Π_k^∞ as

$$V(K) := \{v\in\Pi_k^\infty \mid \mathcal{Y}_K(T,v) \text{ is a nonzero free } \mathcal{A}_p[\langle\tau_{K,v}\rangle]\text{-module}\}$$

and, for every $v\in V(K)$, a natural number

$$r_K(T,v) := \mathrm{rk}_{\mathcal{A}_p}(T(-1)^{\tau_v=1}).$$

(8.2) Example. In the following examples we assume $\mathcal{A}_p$ is the valuation ring $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$.

(i) If $T=\mathcal{O}(1)$, then $V(K)$ is equal to the set of infinite places of k that split in K and $d_K(T,v)=1$ for all $v\in V(K)$.

(ii) If T is self-dual (that is, $T\cong T^*(1)$ as G_k-modules) and $p>2$, then $V(K)=\Pi_k^\infty$ and $d_K(T,v)=(\mathrm{rk}_{\mathcal{O}}T)/2$ for all $v\in V(K)$. To justify this, we write $\rho\colon G_k\to\mathrm{GL}_{\mathrm{rk}_{\mathcal{O}}T}(\mathcal{O})$ for a realisation of the representation T and note self-duality implies $M\rho(\tau_v)M^{-1}=-\rho(\tau_v^{-1})^{\mathrm{t}}$ for some $M\in\mathrm{GL}_{\mathrm{rk}_{\mathcal{O}}T}(\mathcal{O})$. Now, $\tau_v=\tau_v^{-1}$ since $\tau_v^2=\mathrm{id}$ and so we deduce that $\rho(\tau_v)$ has vanishing trace. This implies that 1 and -1 appear with equal multiplicity among the eigenvalues of $\rho(\tau_v)$, and hence that $V(K)=\Pi_k^\infty$. Indeed, if $v\in\Pi_k^\infty$ splits in K, then $v\in V(K)$ if and only if 1 is an eigenvalue of $\rho(\tau_v)$. If v does not split, on the other hand, then $v\in V(K)$ if and only if $\mathrm{rk}_{\mathcal{O}}(T(-1)^{\tau_v=1})=\mathrm{rk}_{\mathcal{O}}(T(-1)^{\tau_v=-1})$.

Assume p satisfies (3.1). Then, with H_v denoting the fixed field of $\tau_{K,v}$ in K, the $\mathcal{A}_p[\mathcal{G}_K]$-module

$$\mathcal{Y}_K(T) := \bigoplus\nolimits_{v\in V(K)}\mathrm{Ind}_{G_k}^{G_{H_v}}(\mathcal{Y}_K(T,v)) \tag{8.3}$$

is a free quotient of $Y_{k,\Pi_k^\infty}(T_{K/k})$ of rank

$$r_K(T) := \sum\nolimits_{v\in V(K)} d_K(T,v). \tag{8.4}$$

We define $e_{K,S,T}$ to be the sum of all primitive idempotents in $A[\mathcal{G}_K]$ that annihilate each of $H^0(\mathcal{O}_{K,S},V_p(M))$, $H^2(\mathcal{O}_{K,S},V_p(M))$ and the kernel of the projection $Y_{k,\Pi_k^\infty}(T_{K/k})\to\mathcal{Y}_K(T)$. We also write $\mathcal{Q}_K$ and $\mathcal{Q}_{K,S,T}$ for the semisimple $\mathbb{C}_p$-algebras $(\mathbb{C}_p\otimes_{\mathbb{Q}}A)[\mathcal{G}_K]$ and $\mathcal{Q}_Ke_{K,S,T}$. Then the descriptions in Proposition 3.45 (ii) combine with Proposition 3.45 (i) to imply the following: the $\mathcal{Q}_{K,S,T}$-module

$$e_{K,S,T}(\mathbb{C}_p\cdot H^1(\mathcal{O}_{k,S(K)},T_{K/k})) = e_{K,S,T}(\mathbb{C}_p\cdot H^1(\mathcal{O}_{K,S(K)},T))$$

is free of rank $r_K(T)$ and any choice of $\mathcal{A}_p[\mathcal{G}_K]$-basis $b_K=\{b_{K,i}\}_{i\in[r_K(T)]}$ of $\mathcal{Y}_K(T)$ induces a composite homomorphism of $\mathcal{Q}_K$-modules

$$\begin{aligned}\Theta_{b_K}\colon \mathbb{C}_p\cdot\mathrm{Det}_{\mathcal{A}_p[\mathcal{G}_K]}(C_{S(K)}(T_{K/k})) &\xrightarrow{\simeq} \mathrm{Det}_{\mathcal{Q}_K}(\mathbb{C}_p\cdot H^1(\mathcal{O}_{k,S(K)},T_{K/k}))\\ &\qquad\otimes_{\mathcal{Q}_K}\mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_p\cdot H^1(C_{S(K)}(T_{K/k})))^{-1} \qquad (8.5)\\ &\longrightarrow e_{K,S,T}\big(\mathbb{C}_p\cdot{\textstyle\bigwedge}_{\mathcal{A}_p[\mathcal{G}_K]}^{r_K(T)} H^1(\mathcal{O}_{K,S(K)},T)\big). \qquad (8.6)\end{aligned}$$

Here the first map is the canonical 'passage-to-cohomology map' and the second is obtained by multiplying by $e_{K,S,T}$ and then evaluating elements of

$$e_{K,S,T}\big(\mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_p\cdot H^1(C_{S(K)}(T_{K/k})))^{-1}\big) = e_{K,S,T}\big(\mathbb{C}_p\cdot{\textstyle\bigwedge}_{\mathcal{A}_p[\mathcal{G}_K]}^{r}\mathcal{Y}_K(T)\big)^{-1}$$

at $e_{K,S,T}\bigwedge_{i\in[r_K(T)]}b_{K,i}$. We then write M_K for the motive $M\otimes_{\mathbb{Q}}h^0(\mathrm{Spec}(K))$, regarded as defined over k and with coefficients $A[\mathcal{G}_K]$, and set

$$\eta_{b_K}\coloneqq\Theta_{b_K}\big((\vartheta^{-1}_{M_K,S(K)}\circ\lambda_{M_K})(L^*_{S(K)}(M^*_K(1),0))\big)\in\mathbb{C}_p\cdot\bigwedge\nolimits^d_{\mathcal{A}_p[\mathcal{G}_K]}H^1(\mathcal{O}_{K,S(K)},T).$$

As in § 4.1, we next fix an abelian pro-p extension $\mathcal{K}$ of k in which all places in Π_k^∞ split completely, write Ω for the collection of finite extensions of k inside $\mathcal{K}$ and consider the family

$$\mathfrak{F}_{\mathrm{rel}}=(\mathscr{F}_{\mathrm{rel}}(T_{K/k},S(K)))_{K\in\Omega}$$

of relaxed Nekovář structures.

Then, assuming the rank $r=r_K(T)$ defined above to be independent of the choice of field K in Ω, the result of Theorem 4.20 can be used to study $\mathrm{TNC}(M,\mathcal{A}_p)$ in the following way.

(8.7) Strategy.

(i) Identify a compatible choice of bases $\{b_K\}_{K\in\Omega}$ such that the family $\eta_{T,\mathcal{A}_p}\coloneqq(\eta_{b_K})_{K\in\Omega}$ defined above belongs to $\mathrm{ES}^r(\mathfrak{F}_{\mathrm{rel}})$.

(ii) Apply Theorem 4.20 to the family $\eta_{T,\mathcal{A}_p}$ and then derive from the conclusion of this result consequences related to $\mathrm{TNC}(M,\mathcal{A}_p)$.

(8.8) Remark. There are perhaps good reasons to expect the existence of a family of bases $\{b_K\}_{K\in\Omega}$ with the property stated in 8.7 (i) in many (and possibly even all) cases of interest. In subsequent sections, we provide details in special cases.

8.2. Statement of the main result

In this section we explain how the theory of Euler systems developed in Part I of the present article allows one to study an Iwasawa-theoretic variant of Conjecture 8.1. In § 10 and § 9 this will be used to obtain results towards Conjecture 8.1 itself in interesting new cases.

Let $\mathcal{O}$ be the ring of integers in a finite extension Φ of $\mathbb{Q}_p$ and with residue field $\kappa\coloneqq\mathcal{O}/(\pi)$, and let T be a finitely generated free $\mathcal{O}$-module endowed with a continuous action of G_k that is unramified outside a finite set $S_{\mathrm{ram}}(T)$ of places of k.

As in § 4.1, we fix a finite subset $S_0\subseteq\Pi_k$ with $\Pi_k^\infty\cup\Pi_k^p\subseteq S_0$ and, for every finite abelian extension K of k set

$$S_0(K)\coloneqq S_0\cup S_{\mathrm{ram}}(K/k)\quad\text{ and }\quad S(K)\coloneqq S_0(K)\cup S_{\mathrm{ram}}(T).$$

Given a subset Σ of Π_k with $\Sigma\cap(S_0\cup S_{\mathrm{ram}}(T))=\varnothing$, we write $\mathcal{K}^\Sigma$ for the composite of all abelian extensions of k which are unramified at Σ, and denote by $\Omega_\Sigma\coloneqq\Omega(\mathcal{K}^\Sigma)$ the set of all finite subextensions of $\mathcal{K}^\Sigma/k$. We abbreviate $\Omega_\varnothing$ to Ω.

For $K\in\Omega_\Sigma$, we write $\mathscr{F}_{\mathrm{rel},\Sigma}(T_{K/k})$ for the 'Σ-modification' $\mathscr{F}_{\mathrm{rel}}(T_{K/k},S(K))_\Sigma$ (cf. Example 3.16) and, for each $i\in\mathbb{N}_0$, we then set

$$H^i_\Sigma(\mathcal{O}_{K,S(K)},T)\coloneqq H^1_{\mathscr{F}_{\mathrm{rel},\Sigma}(T_{K/k})}(k,T_{K/k}).$$

We observe that this construction agrees with previous definitions of Σ-modified cohomology (as used, for example, in [27, § 2.3]), and is motivated by constructions of Gross [47] and Rubin [91] in the context of refinements of Stark's conjectures.

(8.9) Definition. *For each $r\in\mathbb{N}_0$ and finite subset Σ of Π_k with $\Sigma\cap(S_0\cup S_{\mathrm{ram}}(T))=\varnothing$, we define $\mathrm{ES}^r_{\Sigma,S_0}(T)$ to be the set of all elements*

$$c=(c_K)_{K\in\Omega_\Sigma}\in\prod\nolimits_{K\in\Omega_\Sigma}\big(\bigcap\nolimits^r_{\mathcal{O}[\mathcal{G}_K]}H^1_\Sigma(\mathcal{O}_{K,S(K)},T)\big)$$

that satisfy the following distribution relation: for all $K,L\in\Omega_\Sigma$, with $K\subseteq L$ one has

$$\mathrm{cores}^r_{L/K}(c_L)=\big(\prod\nolimits_{v\in S_0(L)\setminus S_0(K)}\mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\big)\cdot c_K\quad\text{ in }\bigcap\nolimits^r_{\mathcal{O}[\mathcal{G}_K]}H^1_\Sigma(\mathcal{O}_{K,S(L)},T).$$

Here $\mathrm{cores}^r_{L/K}$ denotes the morphism that is induced, via Lemma 2.17 (i), by the corestriction map $\mathrm{cores}_{L/K}\colon H^1_\Sigma(\mathcal{O}_{K,S(K)},T)\to H^1_\Sigma(\mathcal{O}_{K,S(L)},T)$ (cf. [12, § 3.2] for details).

(8.10) Remark. The link between the above notion of Euler system and that defined in § 4.1 will be explained in § 8.3.1.

We now fix a finite abelian extension K of k and a $\mathbb{Z}_p$-power extension k_∞ of k in which no finite place splits completely, and set $K_\infty := K \cdot k_\infty$. We set $\Lambda_K := \mathcal{O}[\![\mathcal{G}_{K_\infty}]\!]$ and write $\mathcal{Q}_K$ for its total quotient ring. We also use the fields

$$l_\infty := k(1) \cdot k_\infty(\mu_{p^\infty}, (\mathcal{O}_k^\times)^{1/p^\infty}) \text{ and } l_\infty(T) := (k^c)^{\ker(\rho_T)}$$

with ρ_T the canonical homomorphism $G_{l_\infty} \to \mathrm{Aut}(T)$ (so that $l_\infty \subseteq l_\infty(T)$). Finally, we fix a natural number r (which is to be specified later on) and consider the following hypotheses on the data (T, k_∞, r).

(8.11) Hypotheses. The following conditions are valid.

(i) The $\kappa[G_k]$-module $\overline{T} := T \otimes_{\mathcal{O}} \kappa$ is irreducible.

(ii) There exists $\tau \in G_{l_\infty}$ such that $\dim_{\Bbbk}(\overline{T}/(\tau - 1)\overline{T}) = 1$.

(ii*) If $p = 2$, then $\dim_{\Bbbk}(\overline{T}) = 1$.

(iii) The module $H^1(l_\infty(T)/k, \overline{T}^*(1))$ vanishes.

(iv) If $p \in \{2, 3\}$, then the $\mathbb{Z}_p[G_k]$-modules $\overline{T}$ and $\overline{T}^*(1)$ have no nonzero isomorphic subquotients.

(v) $H^1(l_\infty(T)/k, \overline{T})$ is a finitely generated $\Bbbk$-vector space and

$$\dim_{\Bbbk}\big(H^1(l_\infty(T)/k, \overline{T})\big) < r + \dim_{\Bbbk}\big(H^0(k, \overline{T})\big) - \sum\nolimits_{v \in \Sigma} \dim_{\Bbbk}\big(H^0(k_v, \overline{T}^*(1))\big).$$

(8.12) Remark. If the $\kappa[G_k]$-module $\overline{T}$ is irreducible, then so is $\overline{T}^*(1)$. In particular, the validity of Hypotheses 8.11 (i) and (iii) combine to imply that the modules $H^0(k, \overline{T})$ and $H^0(k, \overline{T}^*(1))$ must both vanish.

We consider the induced representation

$$\mathcal{T} := \mathrm{Ind}_{G_k}^{G_{K_\infty}}(T) = T \otimes_{\mathcal{O}} \Lambda_K,$$

on which $\sigma \in G_k$ acts via $\sigma \cdot (a \otimes b) := (\sigma a) \otimes (b \overline{\sigma}^{-1})$ with $\overline{\sigma}$ the image of σ in Λ_K.
Write $\mathscr{F}_{\mathrm{rel},\Sigma}(\mathcal{T})$ for the Σ-modification of the relaxed Nekovář structure $\mathscr{F}_{\mathrm{rel}}(\mathcal{T}, S(K))$ on $\mathcal{T}$, and denote by $\mathcal{F}_{\mathrm{rel},\Sigma}(\mathcal{T}) := h(\mathscr{F}_{\mathrm{rel},\Sigma}(\mathcal{T}))$ the induced Mazur–Rubin structure. We will also use the complex

$$C_{S(K),\Sigma}(\mathcal{T}) := C(\mathscr{F}_{\mathrm{rel},\Sigma}(\mathcal{T}))$$

from Proposition 3.45. (If $\Sigma = \varnothing$, then we will drop the subscript Σ.) If we further assume that $H^0_\Sigma(\mathcal{O}_{k,S(K)}, T)$ is $\mathcal{O}$-torsion free, then we may apply Lemma 2.35 to the complex $C_{S(K),\Sigma}(\mathcal{T})$ to deduce it admits a resolution of the form $P \to P$ with P a finitely generated free Λ_K-module and the first term placed in degree zero. This combines with [95, Lem. B.9] to imply that, for every integer $r \geq 0$, the natural map

$$\begin{aligned} \bigcap\nolimits^r_{\Lambda_K} H^1_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}) &\to \varprojlim\nolimits_{K \subseteq E \subseteq K_\infty} \bigcap\nolimits^r_{\mathcal{O}[\mathcal{G}_E]} H^1_\Sigma(\mathcal{O}_{k,S(K)}, T \otimes_{\mathcal{O}} \mathcal{O}[\mathcal{G}_E]) \\ &\xrightarrow{\simeq} \varprojlim\nolimits_{K \subseteq E \subseteq K_\infty} \bigcap\nolimits^r_{\mathcal{O}[\mathcal{G}_E]} H^1_\Sigma(\mathcal{O}_{E,S(K)}, T) \end{aligned}$$

is an isomorphism. (Here the second map is the isomorphism from Shapiro's lemma.) For every $c \in \mathrm{ES}^r_{\Sigma,S_0}(T)$ we therefore obtain a well-defined element

$$c_{K_\infty} := (c_E)_E \in \bigcap\nolimits^r_{\Lambda_K} H^1_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}).$$

We next fix a Λ_K-free quotient Y of $Y_{\Pi_k^\infty}(\mathcal{T})$, for example of the form obtained by passing to the limit over the modules $\mathcal{Y}_{EK}(T)$ defined in (8.3). Note that by the middle exact sequence in Proposition 3.45 (v), the module Y is then also a free quotient of $H^1(C_{S,\Sigma}(\mathcal{T}))$. We set

$$r := \mathrm{rk}_{\Lambda_K}(Y)$$

and we assume throughout the sequel that $r > 0$ (or equivalently, that Y is non-zero).
We fix a splitting $\mathcal{G}_{K_\infty} \cong \Delta_K \times \Gamma_K$ with $\Gamma_K \cong \mathbb{Z}_p^n$ for some $n > 0$ and Δ a finite group, we also obtain an identification $\Lambda_K \cong \mathcal{O}[\Delta_K][\![\Gamma_K]\!]$. Given this, we may regard elements of $\Phi[\Delta_K]$ as elements of the total ring of fractions $\mathcal{Q}_K := \mathcal{Q}(\Lambda_K)$ of Λ_K. With Φ^c a choice of algebraic closure of Φ, we then obtain an idempotent of $\Phi[\Delta_K]$ by setting

$$e_K := \sum\nolimits_\psi e_\psi$$

where ψ runs over all characters $\Delta_K \to \Phi^{c,\times}$ for which the primitive idempotent e_ψ annihilates the kernel of the projection $\mathcal{Q}_K \otimes_{\Lambda_K} H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}) \to \mathcal{Q}_K \otimes_{\Lambda_K} Y$.

(8.13) Remark. Since no non-archimedean place of k splits completely in k_∞, the idempotent e_K does not depend on Σ (cf. [12, Lem. 4.12]). In addition, if k_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k, then the weak Leopoldt conjecture (from [86, § 1.3]) predicts that $\mathcal{Q}_K \otimes_{\Lambda_K} H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T})$ vanishes and so $e_K = 1$.

We also define a further idempotent of $\mathcal{Q}_K$ as the sum

$$\epsilon_K := \sum\nolimits_\psi e_\psi$$

over all characters $\psi \colon \Delta_K \to \Phi^{c,\times}$ such that e_ψ annihilates the kernel of the projection map $\mathbb{Q}_K \otimes_{\Lambda_K} Y_{\Pi_k^\infty}(\mathcal{T}) \to \mathbb{Q}_K \otimes_{\Lambda_K} Y$. In particular, this definition ensures that $\mathcal{Q}_K\epsilon_K$ is the kernel of the latter projection map.
Fix a Λ_K-basis $b_\bullet = \{b_i\}_{i\in[r]}$ of Y. Then, since $\mathcal{Q}_K$ is a semisimple ring, there exists a composite morphism of $\mathcal{Q}_K$-modules

$$\begin{aligned}\Theta_{K_\infty,\Sigma,b_\bullet} \colon \mathcal{Q}_K \otimes_{\Lambda_K} \mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T})) &\xrightarrow{\sim} \mathrm{Det}_{\mathcal{Q}_K}(\mathcal{Q}_K \otimes_{\Lambda_K} H^0(C_{S(K),\Sigma}(\mathcal{T})))\\ &\qquad \otimes_{\mathcal{Q}_K} \mathrm{Det}_{\mathcal{Q}_K}(\mathcal{Q}_K \otimes_{\Lambda_K} H^1(C_{S(K),\Sigma}(\mathcal{T})))^{-1}\\ &\to e_K\epsilon_K\mathcal{Q}_K \otimes_{\Lambda_K} \bigwedge\nolimits^r_{\Lambda_K} H^1_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}),\end{aligned} \tag{8.14}$$

in which the first map is the canonical 'passage-to-cohomology map' and the second is induced by first multiplying by $e_K\epsilon_K$ and then evaluating elements of

$$e_K\epsilon_K\big(\mathrm{Det}_{\mathcal{Q}_K}(\mathcal{Q}_K \otimes_{\Lambda_K} H^1(C_{S(K),\Sigma}(\mathcal{T})))^{-1}\big) = e_K\big(\mathcal{Q}_K \otimes_{\Lambda_K} \mathrm{Det}_{\Lambda_K}(Y)^{-1}\big)$$

on $e_K \bigwedge_{i\in[r]} b_i$. Our next result concerns the image under $\Theta_{K_\infty,\Sigma,b_\bullet}$ of the Λ_K-submodule $\mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T}))$ of $\mathcal{Q}_K \otimes_{\Lambda_K} \mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T}))$.
To state it, we use the canonical direct product decomposition

$$\Delta_K = \nabla_K \times \square_K \tag{8.15}$$

in which $\square_K$ and ∇_K are respectively the p-Sylow subgroup of Δ_K and the maximal subgroup of Δ_K or order prime to p. We also set $L := K^{\square_K}$ and $F := K^{\nabla_K}$. Given a character $\chi \colon \nabla_K \to \Phi^{c,\times}$, we write $\mathcal{O}_\chi := \mathcal{O}[\mathrm{im}\,\chi]$ for the $\mathcal{O}$-algebra generated by the values of χ and note that any $\mathcal{O}[\nabla_K]$-module M decomposes as

$$M = \bigoplus\nolimits_{\chi\in\widehat{\nabla_K}/\sim} M_\chi$$

with $M_\chi := M \otimes_{\mathcal{O}[\nabla_K]} \mathcal{O}_\chi$. Given an element $m \in M$, we denote by m_χ the image of m in M_χ. Lastly, we write $T(\chi)$ for the module $T \otimes_{\mathcal{O}} \mathcal{O}_\chi$ with 'χ-twisted' G_k-action given by $\sigma \cdot a := \chi(\sigma)^{-1} \cdot (\sigma a)$ for all $a \in T$.

(8.16) Theorem. *Fix a finite abelian extension K of k and a character $\chi \colon \nabla_K \to \Phi^{c,\times}$. Set $\mathcal{T} := T \otimes_{\mathcal{O}} \Lambda_K$, fix a non-zero free Λ_K-module quotient Y of $Y_{\Pi_k^\infty}(\mathcal{T})$ of rank r, and assume that all of the following conditions are valid.*

(a) Every $v \in S_{\mathrm{ram}}(T^\vee(1))\backslash S_0$ is finitely decomposed in k_∞ and such that $\bigoplus_{w|v} H^0(E_w, T^*(1))$ vanishes for every intermediate field E of K_∞/k.

(b) $H^1_\Sigma(\mathcal{O}_{K,S(K)}, T)$ is $\mathcal{O}$-torsion free.

(c) Hypotheses 8.11 on $(T(\chi), k_\infty, r)$ are valid.

Then, for every $c \in \mathrm{ES}^r_{\Sigma,S_0}(T)$, the following claims are valid.

(i) For each Λ_K-basis $b_\bullet = \{b_i\}_{i\in[r]}$ of Y one has

$$\mathrm{Fitt}^r_{\Lambda_K}(X_{S_0}(\mathcal{T}))\cdot\Big(\prod\nolimits_{v\in S(K)\setminus S_0(K)}\mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\Big)\cdot(c_{K_\infty})_\chi \subseteq \Theta_{K_\infty,\Sigma,b_\bullet}(\mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T})))_\chi.$$

(ii) If $\mathfrak{p}$ is a prime of $\mathcal{Q}_K$ in the support of $\mathcal{Q}_K\otimes_{\Lambda_K} H^1_{\mathcal{F}_{\mathrm{rel},\Sigma}(\mathcal{T})^\vee}(k,\mathcal{T}^\vee(1))^\vee_\chi$, then $(\epsilon_K c_{K_\infty})_\mathfrak{p} = 0$.

(iii) If p satisfies condition (3.1), then for all $f \in H^1_\Sigma(\mathcal{O}_{k,S(K)},\mathcal{T})^*$ one has

$$\epsilon_K\cdot f(c_{K_\infty})_\chi \in \epsilon_K\,\mathrm{Fitt}^0_{\Lambda_K}(H^1_{\mathcal{F}^\vee_{\mathrm{rel},\Sigma}(\mathcal{T})^\vee}(k,\mathcal{T}^\vee(1))^\vee)^{**}_\chi.$$

The proof of this result will occupy the remainder of this section and is given in § 8.4 after we have established a series of preliminary observations in § 8.3.

We will discuss consequences of Theorem 8.16 towards Kato's conjecture (8.1) in the settings of both elliptic curves and the multiplicative group in § 9 and § 10, respectively.

8.3. Preliminary results

We first make several general observations that will be used in § 8.4 to prove Theorem 8.16.

8.3.1. Twisting Euler systems

Suppose R is an $\mathcal{O}$-algebra endowed with a continuous action of G_k, and write $\psi\colon G_k \to \mathrm{Aut}(R)$ for the induced character. We then define a G_k-module as $T(\psi) \coloneqq T\otimes_\mathcal{O} R$ with $\sigma\in G_k$ acting as $\sigma\cdot(x\otimes y) \coloneqq (\sigma x)\otimes(\sigma^{-1}y)$. As a special case, if $K\in\Omega$ and $\psi_K\colon G_k\to\mathcal{O}[\mathcal{G}_K]$ is the character defined by sending $\sigma\in G_k$ to its image in $\mathcal{G}_K$, then $T(\psi_K)$ agrees with the induced representation $T_{K/k} = \mathrm{Ind}^{G_k}_{G_E}(T)$ introduced before.

(8.17) Lemma. *Assume that R satisfies condition (2.5), and that $\mathrm{im}\,\psi$ is abelian and either finite or a finitely generated $\mathbb{Z}_p$-module. Let $\Sigma\subseteq\Pi_k$ be a finite set with $\Sigma\cap S(k_\psi)=\varnothing$ such that $C_{S(k_\psi),\Sigma}(T_{k_\psi/k})$ has a representative of the form $[P\to P]$ with P a finite-rank free $\mathcal{O}[\![\mathcal{G}_{k_\psi}]\!]$-module placed in degree 0.*

(i) Fix an abelian pro-p extension E of k such that $\mathcal{G}_E$ is a finitely generated $\mathbb{Z}_p$-module, and assume that $\Sigma\cap S(E)=\varnothing$. Then, for every integer $r\geq 0$, there are canonical (ψ-semilinear) morphisms

$$\mathrm{Tw}^r_{E,\psi}\colon \bigcap\nolimits^r_{\mathcal{O}[\![\mathcal{G}_{Ek_\psi}]\!]} H^1_\Sigma(\mathcal{O}_{Ek_\psi,S(Ek_\psi)},T)\to\bigcap\nolimits^r_{R[\![\mathcal{G}_E]\!]} H^1_\Sigma(\mathcal{O}_{E,S(Ek_\psi)},T(\psi)).$$

and

$$\mathrm{Tw}^{\mathrm{det}}_{E,\psi}\colon\ \mathrm{Det}_{\mathcal{O}[\![\mathcal{G}_{Ek_\psi}]\!]}(C_{S(Ek_\psi)}(T_{Ek_\psi/k}))\to\mathrm{Det}_{R[\![\mathcal{G}_E]\!]}(C_{S(Ek_\psi)}(T_{E/k}(\psi)))$$

that have the following properties.

(a) Let Y be a free $\mathcal{O}[\![\mathcal{G}_{Ek_\psi}]\!]$-quotient of $Y_{\Pi_k^\infty}(T_{Ek_\psi/k})$ of rank r with basis $b_\bullet$. Write $b'_\bullet$ for the induced $R[\mathcal{G}_E]$-basis of the free rank-r quotient $Y'\coloneqq Y\otimes_{\mathcal{O}[\![\mathcal{G}_{Ek_\psi}]\!]}R[\![\mathcal{G}_E]\!]$ of $Y_{\Pi_k^\infty}(T_{E/k}(\psi))\cong Y_{\Pi_k^\infty}(T_{Ek_\psi/k})\otimes_{\mathcal{O}[\![\mathcal{G}_{Ek_\psi}]\!]}R[\![\mathcal{G}_E]\!]$. Then the following diagram commutes.

$$\begin{array}{ccc}\mathrm{Det}_{\mathcal{O}[\mathcal{G}_E]}(C_{S(Ek_\psi),\Sigma}(T_{Ek_\psi/k})) & \xrightarrow{\ \vartheta\ } & \bigcap\nolimits^r_{R[\![\mathcal{G}_E]\!]} H^1_\Sigma(\mathcal{O}_{Ek_\psi,S(Ek_\psi)},T)\\ \big\downarrow{\scriptstyle \mathrm{Tw}^{\mathrm{det}}_{E,\psi}} & & \big\downarrow{\scriptstyle \mathrm{Tw}^r_{E,\psi}}\\ \mathrm{Det}_{\mathcal{O}[\mathcal{G}_E]}(C_{S(Ek_\psi),\Sigma}(T_{E/k}(\psi))) & \xrightarrow{\ \vartheta'\ } & \bigcap\nolimits^r_{R[\![\mathcal{G}_E]\!]} H^1_\Sigma(\mathcal{O}_{E,S(Ek_\psi)},T(\psi)).\end{array}$$

Here ϑ and ϑ' are the maps from Lemma 2.37 (i) for the data $(C_{S(Ek_\psi),\Sigma}(T_{Ek_\psi/k}), b_\bullet)$ and $(C_{S(Ek_\psi),\Sigma}(T_{E/k}(\psi)), b'_\bullet)$, respectively.

(b) $\mathrm{Tw}^{\det}_{E,\psi}$ *sends an* $\mathcal{O}[\![\mathcal{G}_{Ek_\psi}]\!]$*-basis to an* $R[\![\mathcal{G}_E]\!]$*-basis.*

(ii) *Write* $\mathcal{K}^{\Sigma,p}$ *for the composite of all p-power degree extensions of k contained in* $\mathcal{K}^{\Sigma}$*, and let* $\Omega(\mathcal{K}^{\Sigma,p}) \subseteq \Omega_\Sigma$ *denote the subset of fields contained in* $\mathcal{K}^{\Sigma,p}$*. Then* $\mathfrak{F}_{\mathrm{rel},\Sigma}(T(\psi)) \coloneqq (\mathscr{F}_{\mathrm{rel},\Sigma}(T(\psi)_{E/k}))_{E\in\Omega(\mathcal{K}^{\Sigma,p})}$ *satisfies Hypothesis 4.1 and, moreover, there exists a* ψ*-semilinear map of* $\mathcal{O}[\![\mathcal{G}_k]\!]$*-modules*

$$\mathrm{ES}^r_{\Sigma,S_0}(T) \to \mathrm{ES}^r_{S_0}(\mathfrak{F}_{\mathrm{rel},\Sigma}(T(\psi))), \quad c \mapsto (\mathrm{Tw}^r_{E,\psi}(c_{Ek_\psi}))_{E\in\Omega(\mathcal{K}^{\Sigma,p})}.$$

Proof. It follows from Lemma 3.3 that $\mathfrak{F}(T(\psi))$ satisfies Hypothesis 4.1, as claimed in (ii). The remaining claims are proved by the argument of [10, Lem. A.8] and are induced by the isomorphism $C_{S(Ek_\psi),\Sigma}(T_{Ek_\psi/k}) \otimes_{\mathcal{O}[\![\mathcal{G}_{Ek_\psi}]\!]} R[\![\mathcal{G}_E]\!] \cong C_{S(Ek_\psi),\Sigma}(T(\psi)_{E/k})$ that follows from Proposition 3.45 (iii). (See also [11, Lem. 3.9] for (i) and [92, § 2.4 and Ch. 6] for the second claim in (ii).) □

(8.18) Remark. Fix $\chi\colon \Delta_K \to \Phi^{c,\times}$ and write $\mathcal{O}_\chi$ for the extension of $\mathcal{O}$ generated by the values of χ. We then obtain a character $\psi\colon G_k \to \mathcal{O}_\chi[\mathcal{G}_F]$ and the construction of Lemma 8.17 defines a morphism $\mathrm{ES}^r_{\Sigma,S_0}(T) \to \mathrm{ES}^r_{S_0}(\mathfrak{F}_{\mathrm{rel},\Sigma}(T_{F/k}(\chi)))$. This provides a precise link between the notion of Euler system in Definition 8.9 and that used in § 4.1.

8.3.2. Torsion subgroups

The following result clarifies the condition in Theorem 8.16 (b).

(8.19) Lemma. *Suppose Hypotheses 8.11 (i) (ii) and Hypothesis 4.16 (ii) are valid for T (with* $R = \mathcal{O}$*). Then the following assertions are equivalent for every finite set* $\Sigma \subseteq \Pi_k$ *with* $\Sigma\cap S(k) = \varnothing$*, field* $F \in \Omega(\mathcal{K}^{\Sigma,p})$*, and finite set* $U \subseteq \Pi_k$ *with* $U \supseteq S(F)$*.*

(i) $H^0_\Sigma(\mathcal{O}_{k,U}, \overline{T}) \neq 0$*,*

(ii) $H^1_\Sigma(\mathcal{O}_{F,U}, T)$ *is not* $\mathcal{O}$*-torsion free,*

(iii) $\Sigma = \varnothing$*,* $\mathrm{rk}_{\mathcal{O}}(T) = 1$*, and* G_k *acts trivially on* $\overline{T}$*.*

Proof. At the outset we define $V \coloneqq \Phi \otimes_{\mathcal{O}} T$. The long exact sequence in cohomology arising from the short exact sequence $0 \to T \to V \to V/T \to 0$ then combines with the vanishing of T^{G_k} (that follows from Hypothesis 4.16 (ii)) to give the exact sequence

$$0 \longrightarrow H^0_\Sigma(\mathcal{O}_{F,U}, V/T) \longrightarrow H^1_\Sigma(\mathcal{O}_{F,U}, T) \longrightarrow H^1_\Sigma(\mathcal{O}_{F,U}, V).$$

Since $H^1_\Sigma(\mathcal{O}_{F,U}, V) = \Phi \otimes_{\mathcal{O}} H^1_\Sigma(\mathcal{O}_{F,U}, T)$ by Proposition 3.45 (iii), we deduce an identification $H^1_\Sigma(\mathcal{O}_{F,U}, T)_{\mathrm{tor}} \cong H^0_\Sigma(\mathcal{O}_{F,U}, V/T)$. The latter is non-trivial if and only if $H^0_\Sigma(\mathcal{O}_{k,U}, V/T) \neq 0$ because F is a p-extension of k (cf. [82, Cor. 1.6.13]). Writing π for a uniformiser of $\mathcal{O}$, we have an isomorphism $(V/T)[\pi] \cong \overline{T}$, hence $H^0_\Sigma(\mathcal{O}_{k,U}, V/T) \neq 0$ is also equivalent to $H^0_\Sigma(\mathcal{O}_{k,U}, \overline{T}) \neq 0$. This proves the equivalence of conditions (ii) and (i).
Now, the triangle (3.17) implies that $H^0_\Sigma(\mathcal{O}_{k,S}, \overline{T})$ is equal to the kernel of the diagonal map $H^0(k, \overline{T}) \to \bigoplus_{v\in\Sigma} H^0(k_v, \overline{T})$. Since the latter is injective as soon $\Sigma \neq \varnothing$, we conclude that (i) implies $\Sigma = \varnothing$. In addition, $\overline{T}$ is an irreducible $\Bbbk[G_k]$-module by Hypothesis 8.11 (ii), and so $H^0(\mathcal{O}_{k,U}, \overline{T})$ must be equal to $\overline{T}$ if it is nonzero. By Hypothesis 8.11 (i) we then have that $\overline{T} = \overline{T}^{\tau=\mathrm{id}}$ is a one-dimensional $\Bbbk$-vector space, which implies that $\mathrm{rk}_{\mathcal{O}}(T) = 1$ by Nakayama's lemma. This shows that (i) implies (iii), and since (iii) clearly implies (i), this proves the lemma. □

8.3.3. Core ranks

We next clarify Hypothesis 4.14 (vi) in the case of relaxed Nekovář structures. To do this, we use the notation specified in (5.1). In particular, we now write Λ and A for R_i and $\mathcal{T} \otimes_{\mathcal{R}} \Lambda$, so that $A \otimes_\Lambda \Bbbk = \overline{T}$.

We fix finite subsets S and Σ of Π_k with $S(k) \subseteq S$ and $\Sigma \cap S = \varnothing$ and recall that, if p verifies (3.1), then the Σ-modified relaxed Nekovář structure $\mathscr{F}_{\mathrm{rel}}(A,S)_\Sigma$ on A coincides with both of the induced structures $\mathscr{F}_{\mathrm{rel}}(\mathcal{T},S)_\Sigma \otimes_{\mathcal{R}} \Lambda$ and $\mathscr{F}_{\mathrm{rel}}(\mathcal{T}\otimes_{\mathcal{R}}\mathcal{R}_i,S)_\Sigma \otimes_{\mathcal{R}_i} \Lambda$ (cf. Example 3.18 (iv)). We also recall that a finitely generated Λ-module M, with $m \coloneqq \dim_{\Bbbk}(M \otimes_\Lambda \Bbbk)$, is said to be 'quadratically-presented' if there exists an exact sequence of Λ-modules of the form

$$\Lambda^{\oplus m} \to \Lambda^{\oplus m} \to M \to 0.$$

(8.20) Lemma. *Fix finite subsets S and Σ of Π_k with $S(k) \subseteq S$ and $\Sigma \cap S = \varnothing$. Write $\tilde{F}$ for the Mazur–Rubin structure $h(\mathscr{F}_{\mathrm{rel}}(A,S)_\Sigma)$ on A and $\overline{F}$ for the structure that it induces on $\overline{T}$. Then, if p verifies (3.1), the following claims are valid.*

(i) The integer $\chi(\overline{F},j)$ is equal to

$$\dim_{\Bbbk}\big(H^0(k,\overline{T})\big) + \dim_{\Bbbk}\big(X_S(\overline{T})\big) + \dim_{\Bbbk}\big(Ш_{\overline{F}^*,j}(\overline{T}^*(1))\big) - \dim_{\Bbbk}\big(Ш_{\overline{F},j}(\overline{T})\big) + \sum\nolimits_{v\in\Sigma} \dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big) + \sum\nolimits_{v\in S\setminus\Pi_k^\infty} \dim_{\Bbbk}\big(\mathrm{Tor}_1^\Lambda(H^0(k_v,A^*(1))^*,\Bbbk)\big).$$

(ii) If the Λ-module $H^2(k_v,A)$ is quadratically presented for every $v \in S \setminus \Pi_k^\infty$, then $\chi(\overline{F},j)$ is at least

$$\sum\nolimits_{v\in\Pi_k^\infty} \dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big) + \dim_{\Bbbk}\big(Ш_{\overline{F}^*,j}(\overline{T}^*(1))\big) - \dim_{\Bbbk}\big(Ш_{\overline{F},j}(\overline{T})\big) + \dim_{\Bbbk}\big(H^0(k,\overline{T})\big) - \dim_{\Bbbk}\big(H^0(k,\overline{T}^*(1))\big) - \sum\nolimits_{v\in\Sigma} \dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big).$$

Proof. We write F_0 for the Mazur–Rubin structure $h(\mathscr{F}_{\mathrm{rel}}(A,S)_\Sigma \otimes_\Lambda \Bbbk)$ on $\overline{T}$. Then, by using Theorem 3.43 (ii) to compare F_0 with $\overline{F}$, one finds that

$$\begin{aligned}\chi(\overline{F},j) =& \chi(F_0,j) - \sum\nolimits_{v\in S(\mathcal{F})} \dim_{\Bbbk}\big(H^1_{/\overline{F}}(k_v,\overline{T})\big)\\ =& \dim_{\Bbbk}\big(H^0(k,\overline{T})\big) + \dim_{\Bbbk}\big(X_{S(\mathcal{F})}(\overline{T})\big) - \sum\nolimits_{v\in S(\mathcal{F})} \dim_{\Bbbk}\big(H^1_{/\overline{F}}(k_v,\overline{T})\big)\\ &+ \sum\nolimits_{v\in\Sigma} \dim_{\Bbbk}\big(H^0(k,\overline{T})\big) + \dim_{\Bbbk}\big(Ш_{\overline{F}^*,j}(\overline{T}^*(1))\big) - \dim_{\Bbbk}\big(Ш_{\overline{F},j}(\overline{T})\big).\end{aligned}$$

In this computation we have also used Proposition 3.45 (v) and the equality $\chi_{\Bbbk}(C_{S,\Sigma}(\overline{T})) = 0$ that follows from Proposition 3.45 (i) and Lemma 3.3.
Now, by the definition of induced Mazur–Rubin structure, one has

$$H^1_{/\overline{F}}(k_v,\overline{T}) \coloneqq \mathrm{coker}\,\big\{H^1_{\tilde{F}}(k_v,A) \subseteq H^1(k_v,A) \to H^1(k_v,\overline{T})\big\}.$$

In addition, for $v \in \Sigma$, one has $H^1_{\tilde{F}}(k_v,A) = 0$, so $H^1_{/\overline{F}}(k_v,\overline{T}) = H^1(k_v,\overline{T})$ and hence

$$\begin{aligned}\dim_{\Bbbk}\big(H^1_{/\overline{F}}(k_v,\overline{T})\big) &= \dim_{\Bbbk}\big(H^1(k_v,\overline{T})\big)\\ &= \dim_{\Bbbk}\big(H^0(k_v,\overline{T})\big) + \dim_{\Bbbk}\big(H^2(k_v,\overline{T})\big)\\ &= \dim_{\Bbbk}\big(H^0(k_v,\overline{T})\big) + \dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big).\end{aligned}$$

It remains to consider the case $v \in S$. In this case $H^1_{/\overline{F}}(k,\overline{T})$ is defined to be the cokernel of the natural map $\rho_v : H^1(k_v,A) \to H^1(k_v,\overline{T})$. In addition, the spectral sequence

$$E_2^{i,j} = \mathrm{Tor}^\Lambda_{-i}(H^j(k_v,A),\Bbbk) \Rightarrow E^{i+j} = H^{i+j}(k_v,\overline{T})$$

induces an the exact sequence

$$\mathrm{Tor}_2^\Lambda(H^2(k_v,A),\Bbbk) \to H^1(k_v,A)\otimes_\Lambda \Bbbk \to H^1(k_v,\overline{T}) \to \mathrm{Tor}_1^\Lambda(H^2(k_v,A),\Bbbk) \to 0.$$

in which the second arrow is induced by ρ_v. In particular, since we are assuming p verifies (3.1), it follows that $H^1_{/\overline{F}}(k_v,\overline{T}) = (0)$ for each $v \in \Pi_k^\infty$. In addition, for $v \in S\setminus\Pi_k^\infty$, the above sequence combines with local duality to imply that $H^1_{/\overline{F}}(k_v,\overline{T})$ is isomorphic to $\mathrm{Tor}_1^\Lambda(H^0(k_v,A^*(1))^*,\Bbbk)$, as required to complete the proof of (i).

Before turning to the proof of (ii), we note that $\dim_{\Bbbk}\big(\mathrm{Tor}_1^{\Lambda}(M,\Bbbk)\big) \le \dim_{\Bbbk}(M\otimes_{\Lambda}\Bbbk)$ for every quadratically presented Λ-module M. To justify this, we fix a presentation

$$\Lambda^{\oplus n} \xrightarrow{f} \Lambda^{\oplus n} \to M \to 0$$

with $n := \dim_{\Bbbk}(M\otimes_{\Lambda}\Bbbk)$. Setting $I := \ker(f)$, we thereby obtain an induced exact sequence

$$0 = \mathrm{Tor}_1^{\Lambda}(\Lambda^{\oplus n},\Bbbk) \to \mathrm{Tor}_1^{\Lambda}(M,\Bbbk) \to I\otimes_{\Lambda}\Bbbk,$$

from which it follows that $\dim_{\Bbbk}(\mathrm{Tor}_1^{\Lambda}(M,\Bbbk)) \le \dim_{\Bbbk}(I\otimes_{\Lambda}\Bbbk) \le n$ since I is a quotient of $\Lambda^{\oplus n}$. This proves the claim.

Then, by applying this observation with $M = H^2(k_v, A) \cong H^0(k_v, A^*(1))^*$ for each $v \in S\setminus\Pi_k^\infty$, and noting $H^2(k_v,A)\otimes_{\Lambda}\Bbbk$ identifies with $H^2(k_v,\overline{T}) \cong H^0(k_v,\overline{T}^*(1))^*$, we obtain an inequality

$$\sum_{v\in S\setminus\Pi_k^\infty}\big(\dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big) - \dim_{\Bbbk}\big(\mathrm{Tor}_1^{\Lambda}(H^0(k_v,A^*(1))^*,\Bbbk)\big)\big) \ge 0.$$

From the definition of $X_S(\overline{T})$ it therefore follows that

$$\begin{aligned}
&\dim_{\Bbbk}\big(X_S(\overline{T})\big) + \dim_{\Bbbk}\big(H^0(k,\overline{T}^*(1))\big)\\
&= \sum_{v\in\Pi_k^\infty}\dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big) + \sum_{v\in S\setminus\Pi_k^\infty}\dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big)\\
&\ge \sum_{v\in\Pi_k^\infty}\dim_{\Bbbk}\big(H^0(k_v,\overline{T}^*(1))\big) + \sum_{v\in S\setminus\Pi_k^\infty}\dim_{\Bbbk}\big(\mathrm{Tor}_1^{\Lambda}(H^0(k_v,A^*(1))^*,\Bbbk)\big).
\end{aligned}$$

The result of (ii) is now obtained by substituting this inequality into the formula of (i). □

(8.21) Remark. The condition of quadratic-presentability is automatic in the following cases.

(i) Let $\Lambda := (\mathbb{Z}/p^i\mathbb{Z})[G]$ with $i\in\mathbb{N}$ and G a finite abelian group. Then any G-cohomologically trivial finitely generated Λ-module is quadratically presented.

(ii) If Λ is a principal ideal ring, then every Λ-module is quadratically presented. Indeed, as a local principal ideal ring, Λ is isomorphic to a quotient of a discrete valuation ring Λ' and since the claim is true over Λ', base-changing to Λ gives the desired presentation.

8.3.4. Localisation, Euler factors, and Fitting ideals

We recall (from [23, § 3C1]) that a prime $\mathfrak{p}$ in $\mathrm{Spec}^1(\Lambda_K)$ is said to be 'regular' if it does not contain the order of the torsion subgroup of $\mathcal{G}_{K_\infty}$. For such $\mathfrak{p}$, the localisation $\Lambda_{K,\mathfrak{p}}$ of Λ_K is a discrete valuation ring and there exists a character $\chi = \chi_{\mathfrak{p}}\colon \Delta_K \to \Phi^{c,\times}$ and a height-one prime $\wp_\chi$ of $\Lambda_\chi := \mathcal{O}_\chi[\![\Gamma_K]\!]$ for which there is an identification $\Lambda_{K,\mathfrak{p}} = \Lambda_{\chi,\wp_\chi}$.

If $\mathfrak{p}$ is not regular, then it is said to be 'singular'. If $\mathfrak{p}$ is any such prime, then $p\in\mathfrak{p}$, and there exists a character $\chi\colon \nabla_K \to \Phi^{c,\times}$ together with a height-one prime ideal $\wp_\chi$ of $(\Lambda_K)_\chi := \Lambda_\chi[\square_K]$ such that $\Lambda_{K,\mathfrak{p}} = (\Lambda_K)_{\chi,\wp_\chi}$. There is then also the following useful localisation criterion from [20, Lem. 6.3] (see also [40, Lem. 5.6]).

(8.22) Lemma. *Fix a character $\chi\colon\nabla_K\to\Phi^{c,\times}$ and a prime $\wp\in\mathrm{Spec}^1(\Lambda_\chi[\square_K])$ with $p\in\wp$. Then for every finitely generated torsion $\Lambda_\chi[\square_K]$-module with vanishing Iwasawa μ-invariant (as a Λ_χ-module), one has $M_\wp = 0$.*

The following result is useful in the computation of the ideal $\mathrm{Fitt}^0_{\Lambda_K}(X_S(\mathcal{T}))$.

(8.23) Lemma. *For every $v\in\Pi_k\setminus\Pi_k^\infty$ and $\mathfrak{p}\in\mathrm{Spec}^1(\Lambda_K)$, the following claims are valid.*

(i) Assume $v\notin\Pi_k^p$ and either $v\notin S_{\mathrm{ram}}(T^\vee(1))$ or both v is finitely decomposed in k_∞ and $H^0(k_v,T^*_{E/k}(1)) = (0)$ for all finite extensions E of k in K_∞. Then, if $p\notin\mathfrak{p}$, and we write χ for the character $\chi_{\mathfrak{p}}$ defined above, one has

$$\mathrm{Fitt}^0_{\Lambda_K}(H^0(k_v,\mathcal{T}^\vee(1))^\vee)_{\mathfrak{p}} = \begin{cases}\mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\Lambda_{K_\chi,\wp_\chi} & \text{if } v\notin S_{\mathrm{ram}}(K_\chi/k),\\ \Lambda_{K_\chi,\wp_\chi} & \text{if } v\in S_{\mathrm{ram}}(K_\chi/k).\end{cases}$$

(ii) If v is finitely decomposed in k_∞ and $p\in\mathfrak{p}$, then $H^0(k_v,\mathcal{T}^\vee(1)^\vee)_{\mathfrak{p}} = (0)$.

Proof. To prove (i), we assume that $p \notin \mathfrak{p}$. In this case there are isomorphisms

$$(H^0(k_v, \mathcal{T}^\vee(1))^\vee)_\mathfrak{p} \cong H^2(k_v, \mathcal{T})_\mathfrak{p} \cong H^2(k_v, T_{K_{\chi,\infty}/k})_{\wp_\chi} \cong H^0(k_v, T^\vee_{K_{\chi,\infty}/k}(1))^\vee_{\mathfrak{p}_\chi},$$

and so it suffices to investigate $\mathrm{Fitt}^0_{\Lambda_{K_\chi}}(H^0(k_v, T^\vee_{K_{\chi,\infty}/k}(1))^\vee)$. In the remainder of this argument we will therefore assume that $K = K_\chi$.

To discuss the case $v \in S_{\mathrm{ram}}(K/k)$, we write I for the inertia subgroup of $\mathcal{G}_K$ and regard the trace element N_I as an element of Λ_K in the natural way. By assumption, $\chi(I) \neq 1$ and so in $\Lambda_{K,\mathfrak{p}}$ one has $\mathrm{N}_I = \chi(\mathrm{N}_I) = 0$. On the other hand, N_I acts as multiplication by $|I| \in \Lambda^\times_{K,\mathfrak{p}}$ on $H^0(k_v, \mathcal{T}^\vee(1))^\vee = \bigoplus_{w|v} H^0(K_{\infty,w}, T^\vee(1))^\vee$ and so we conclude that, in this case, the localisation of $H^0(k_v, \mathcal{T}(1))^\vee$ at $\mathfrak{p}$ vanishes. This proves the claim if $v \in S_{\mathrm{ram}}(K/k)$, and so it remains to consider the case $v \notin S_{\mathrm{ram}}(K/k)$.

Let us first assume that, in addition, $v \notin S_{\mathrm{ram}}(T^\vee(1))$ so that the action of G_{k_v} is unramified on $\mathcal{T}^\vee(1)$. Taking Matlis duals of the exact sequence

$$0 \longrightarrow H^0(k_v, \mathcal{T}^\vee(1)) \longrightarrow \mathcal{T}^\vee(1) \xrightarrow{\mathrm{Frob}_v - 1} \mathcal{T}^\vee(1),$$

we obtain an identification $H^0(k_v, \mathcal{T}^\vee(1))^\vee \cong \mathrm{coker}\{\mathcal{T}(-1) \xrightarrow{\mathrm{Frob}_v - 1} \mathcal{T}(-1)\}$. As such, we obtain the required equality via the computation

$$\begin{aligned}\mathrm{Fitt}^0_{\Lambda_K}(H^0(k_v, \mathcal{T}^\vee(1))^\vee) &= \Lambda_K \cdot \det_{\Lambda_K}(\mathrm{Frob}_v - 1 \mid \mathcal{T}(-1)) \\ &= \Lambda_K \cdot \det_{\Lambda_K}(\mathrm{Frob}_v^{-1} - 1 \mid \mathcal{T}^*(1)) \\ &= \Lambda_K \cdot \mathrm{Eul}_v(\mathrm{Frob}_v^{-1}).\end{aligned}$$

We finally assume that v is finitely decomposed in k_∞ and that $H^0(k_v, T^*_{E/k}(1))$ vanishes for all finite extensions of E of k contained in K_∞. Set $V \coloneqq T \otimes_\mathcal{O} \Phi$, then the exact sequence $0 \to T^*(1) \to V^*(1) \to V^*(1)/T^*(1) \to 0$ combines with the assumption to imply the exactness of the sequence

$$0 \longrightarrow H^0(k_v, V^*_{E/k}(1)/T^*_{E/k}(1)) \longrightarrow H^1(k_v, T^*_{E/k}(1)) \longrightarrow H^1(k_v, V^*_{E/k}(1))$$

for every finite extension E of k contained in K_∞. We conclude that one has an identification $H^0(k_v, T^\vee_{E/k}(1)) = H^0(k_v, V^*_{E/k}(1)/T^*_{E/k}(1)) \cong H^1(k_v, T^*_{E/k}(1))_{\mathrm{tor}}$. Writing $\mathbb{F}_v$ for the residue field of k_v and $I_v \subseteq G_{K_v}$ for the inertia subgroup, the inflation-restriction sequence gives

$$0 \longrightarrow H^1(\mathbb{F}_v, T^*_{E/k}(1)^{I_v})_{\mathrm{tor}} \longrightarrow H^1(k_v, T^*_{E/k}(1))_{\mathrm{tor}} \longrightarrow H^1(I_v, T^*_{E/k}(1))^{G_{k_v}/I_v}_{\mathrm{tor}}.$$

In addition, the assumption also implies that we have the exact sequence

$$0 \longrightarrow T^*_{E/k}(1)^{I_v} \xrightarrow{\mathrm{Frob}_v - 1} T^*_{E/k}(1)^{I_v} \longrightarrow H^1(\mathbb{F}_v, T^*_{E/k}(1)^{I_v}) \longrightarrow 0 \tag{8.24}$$

which, by comparing $\mathcal{O}$-ranks, shows that $H^1(\mathbb{F}_v, T^*_{E/k}(1)^{I_v})$ is finite and, in particular, equal to its $\mathcal{O}$-torsion submodule.

We now claim that $H^1(I_v, T^*_{E/k}(1))^{G_{k_v}/I_v}_{\mathrm{tor}}$ is a finite group of cardinality bounded independently of E. Indeed, our assumption that v is finitely decomposed in k_∞ implies that $\bigoplus_{w|v} H^1(I_w, T^*(1))$ is a finitely generated $\mathcal{O}$-module of rank bounded independently of E, and so $H^1(I_v, T^*_{E/k}(1))^{G_{k_v}/I_v}_{\mathrm{tor}}$, which identifies with $(\bigoplus_{w|v} H^1(I_w, T^*(1))_{\mathrm{tor}})^{G_{k_v}/I_v}$ by Shapiro's lemma, is finite of order bounded independently of E.

Passing to the limit over E in the exact sequence (8.24), we then deduce from the previous discussion that

$$H^0(k_v, \mathcal{T}^\vee(1))^\vee_\mathfrak{p} \cong (\varprojlim_E H^1(k_v, T^*_{E/k}(1))^\vee_{\mathrm{tor}})_\mathfrak{p} \cong (\varprojlim_E H^1(\mathbb{F}_v, T^*_{E/k}(1)^{I_v})^\vee)_\mathfrak{p}.$$

Note that $\mathcal{T}^*(1)^{I_v}$ is a free Λ_K-module of finite rank since v is assumed to be unramified in K_∞. Taking $\mathcal{O}$-linear duals of (8.24) and passing to the limit (over E), we obtain an exact sequence

$$0 \longrightarrow (\mathcal{T}^*(1)^{I_v})^* \xrightarrow{\mathrm{Frob}_v - 1} (\mathcal{T}^*(1)^{I_v})^* \longrightarrow \varprojlim_E H^1(\mathbb{F}_v, T^*_{E/k}(1)^{I_v})^\vee \longrightarrow 0$$

which we may use to calculate that

$$\begin{aligned}\mathrm{Fitt}^0_{\Lambda_K}(\varprojlim_E H^1(\mathbb{F}_v, T^*_{E/k}(1)^{I_v})^\vee) &= \Lambda_K \cdot \det_{\Lambda_K}(\mathrm{Frob}_v - 1 \mid (\mathcal{T}^*(1)^{I_v})^*)\\ &= \Lambda_K \cdot \det_{\Lambda_K}(\mathrm{Frob}_v^{-1} - 1 \mid \mathcal{T}^*(1)^{I_v})\\ &= \Lambda_K \cdot \mathrm{Eul}_v(\mathrm{Frob}_v^{-1}),\end{aligned}$$

as required to conclude the proof of (i).
As for (ii), taking the Matlis dual of the inclusion $H^0(K_{\infty,w}, T^\vee(1)) \subseteq T^\vee(1)$ for every $w \mid v$ gives a surjection $\bigoplus_{w|v} T(-1) \twoheadrightarrow \bigoplus_{w|v} H^0(K_{\infty,w}, T^\vee(1))^\vee = H^0(k_v, \mathcal{T}^\vee(1))^\vee$. If v is finitely decomposed in k_∞, then the latter is therefore a finitely generated $\mathcal{O}$-module and so, by Lemma 8.22, vanishes after localising at any prime in $\mathrm{Spec}^1(\Lambda_K)$ that contains p. □

(8.25) Remark. In many arithmetic examples, there is a useful replacement for Lemma 8.23 (i) for p-adic places v. To be precise, we suppose $T = H^i_{\text{ét}}(X_{k^c}, \mathcal{O})(1)$ for an odd integer i and a proper smooth variety X that is defined over k and has potentially good reduction at a p-adic place v. If k_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k, then $H^0(k_v, T^\vee_{K_\infty/k}(1))$ is finite by a result of Kubo and Taguchi [66, Th. 1.1] (if X is an elliptic curve, this is a classical result of Imai [50]). In addition, if X is an abelian variety, $i = 1$, and K_∞ is a Lubin–Tate extension, then criteria for finiteness have also been obtained by Ozeki in [85]. If T is the representation attached to a cuspidal newform, then a similar result is obtained by Kato in [60, (12.5.1)].

8.4. The proof of Theorem 8.16

We first deduce the following intermediate result from Theorem 4.20.

(8.26) Proposition. *Fix $c \in \mathrm{ES}^r_{\Sigma,S_0}(T, \mathcal{K})$. Then, under the hypotheses of Theorem 8.16, one has* $\mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T})) \cdot (c_{K_\infty})_\chi \subseteq \Theta_{K_\infty,\Sigma,b_\bullet}(\mathrm{Det}_{\Lambda_K}(C_{S,\Sigma}(\mathcal{T})))_\chi$.

Proof. Set $\mathcal{T}_\chi \coloneqq T(\chi) \otimes_{\mathcal{O}} \Lambda_K$ and write $\mathrm{Tw}^r \colon \mathrm{ES}^r_{\Sigma,S_0}(T, \mathcal{K}) \to \mathrm{ES}^r_{S_0}(\mathscr{F}_{\mathrm{rel},\Sigma}(\mathcal{T}_\chi))$ for the map arising from Lemma 8.17 (ii). We then aim to apply Theorem 4.20 to the Euler system $\mathrm{Tw}^r(c)$. To do this, we must specify the data for and verify the hypotheses of Theorem 4.20.
Firstly, $\mathcal{K}^{\Sigma,p}$ clearly satisfies Hypothesis 4.16 (i). As for Hypothesis 4.16 (ii), we note that in order to prove the claimed result we may replace $T(\chi)$ and c by $T(\chi\rho)$ and $\mathrm{Tw}^r_\rho(c)$ for a character $\rho \colon \mathcal{G}_{k_\infty} \to \mathbb{Z}_p^\times$ without loss of generality. Indeed, since l_∞ contains k_∞ and $\tau \in G_{k_\infty}$, one has $\rho(\tau) = 1$ so that Hypothesis 8.11 (ii) is unaffected. Moreover, $\mathcal{G}_{k_\infty}$ being pro-p implies that the image of ρ is contained in $1 + p\mathbb{Z}_p$ so that $\rho(\sigma) \equiv 1 \mod p$ for all $\sigma \in \mathcal{G}_{k_\infty}$. As a consequence, $\overline{T(\chi)}$ and $\overline{T(\chi\rho)}$ agree, hence the validity of Hypotheses 8.11 for $T(\chi)$ imply their validity also for $T(\chi\rho)$.
In addition, the commutative square in Lemma 8.17 (i) (a) shows that the conclusion of Proposition 8.26 holds for c if and only if it holds for $\mathrm{Tw}^r_\rho(c)$.
Now [92, Lem. 6.1.3 (i)] ensures the existence of a ρ such that $T(\chi\rho)$ satisfies Hypothesis 4.16 (ii). The latter being a quotient of $\mathcal{T}_\chi(\rho) \coloneqq T(\chi\rho) \otimes_{\mathcal{O}} \Lambda_F$, it follows that also $\mathcal{T}_\chi(\rho)$ satisfies Hypothesis 4.16 (ii).
In the notation of § 7 we now take $\mathcal{R} = \Lambda_F$ so that $\mathbb{K} = \mathcal{O}/(\pi)$. In particular, $\overline{\mathcal{T}_\chi} = \overline{T(\chi)}$ and so the validity of the parts of Hypotheses 4.14 (i) and (iv) that concern $\mathcal{T}_\chi$ follows from the assumed validity of Hypotheses 8.11 for $T(\chi)$. In addition, Hypothesis 4.14 (v) is valid by Lemma 8.19 and the assumption $H^1_\Sigma(\mathcal{O}_{K,S(K)}, T)$ is $\mathcal{O}$-torsion free.
Next we note that the field k_∞ from § 7 is a subfield of l_∞, and the field $k(\mathcal{T}_\chi)_\infty$ from § 7 agrees with $l_\infty(T(\chi))$. Given this, the validity of Hypotheses 4.14 (vii) in this case is clear (cf. Remark 4.15 (vi)) and that of Hypotheses 4.14 (ii) and (iii) follows from the assumed validity

of Hypotheses 8.11 for $T(\chi)$.

It still remains for us to specify the data of a morphism $\varrho\colon \mathcal{R} \to R$ of the form required for an application of Theorem 4.20. Before doing this, however, we observe that the claimed result will follow if we can prove an inclusion

$$\mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T})) \cdot (c_{K_\infty})_\chi \subseteq \vartheta_Y(\mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T})))_\chi. \tag{8.27}$$

Indeed, the claimed inclusion follows directly from this after recalling that the maps ϑ_Y and $\Theta_{K_\infty,\Sigma,b_\bullet}$ coincide as a consequence of [27, Prop. A.11 (ii)]. Further, the Λ_K-module $\vartheta_Y(\mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T})))_\chi$ is reflexive (since $\mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T}))_\chi$ is invertible and hence reflexive) and so Lemma 2.4 (ii) implies it is enough to verify (8.27) after localising at each prime in $\mathrm{Spec}^1((\Lambda_K)_\chi)$. For the remainder of the proof we shall therefore fix $\mathfrak{p} \in \mathrm{Spec}^1((\Lambda_K)_\chi)$. It is then convenient to separate the discussion into two cases.

(1) The case $p \in \mathfrak{p}$

We first claim that k_∞ contains a $\mathbb{Z}_p$-extension k'_∞ of k in which no finite place contained in $S(K)$ splits completely. Writing $\mathcal{G}_v \subseteq \mathcal{G}_{k_\infty}$ for the decomposition group of $v \in S(K)_{\mathrm{fin}}$, it suffices to find a direct summand $H \subseteq \mathcal{G}_{k_\infty}$ of $\mathbb{Z}_p$-corank one that does not contain any of the $\mathcal{G}_v$ for $v \in S(K)_{\mathrm{fin}}$. Indeed, the fixed field $k'_\infty \coloneqq k_\infty^H$ of H then has the required property. By assumption, no finite place splits completely in k_∞ and so each $\mathcal{G}_v$ is a $\mathbb{Z}_p$-module of rank at least one. We can therefore pick a nonzero element $x_v \in \mathcal{G}_v$ for every finite $v \in S(K)$ and will now construct by induction on $0 \leq s < \mathrm{rk}_{\mathbb{Z}_p}(\mathcal{G}_{k_\infty})$ a direct summand $H_s \subseteq \mathcal{G}_{k_\infty}$ of $\mathbb{Z}_p$-rank s that does not contain any of the x_v. If $s = 0$, then we may take $H = (0)$ because all x_v are nonzero. Now assume $s \geq 1$ and the claim is already proved for $s - 1$. Then all x_v are nonzero in the quotient $\mathcal{G}_{k_\infty}/H_{s-1}$. In addition, this quotient has $\mathbb{Z}_p$-rank at least two because $s < \mathrm{rk}_{\mathbb{Z}_p}(\mathcal{G}_{k_\infty})$, and hence contains infinitely many direct summands of $\mathbb{Z}_p$-rank one. Since the x_v are only contained in finitely many of these, we can choose an element $y \in \mathcal{G}_{k_\infty}/H_{s-1}$ such that $\mathbb{Z}_p y$ is a direct summand that does not contain any of the x_v. Given this, we may then define $H_s \coloneqq H_{s-1} + \mathbb{Z}_p y$ to conclude the inductive step.

Writing k'_i for the i-th layer of k'_∞/k, we set $U'_i \coloneqq \mathrm{Gal}(k_\infty/k'_i)$. We can then fix a basis $\{U_i\}_{i \in \mathbb{N}}$ of open neighbourhoods of the identity of $\mathcal{G}_{k_\infty}$ in such a way that $U_i \subseteq U'_i$ for every i. As in Remark 2.7 (iii), we now take $\mathcal{R} \coloneqq \Lambda_K$ and $\mathfrak{a}_i \coloneqq (\pi^i, I(U_i))$ with the augmentation ideal $I(U_i)$ so that $\mathcal{R}_i \cong (\mathcal{O}/(\pi^i))[\Gamma/U_i][\Box_K]$. We further take $R \coloneqq (\mathcal{O}/(\pi))[\![\mathcal{G}_{k'_\infty}]\!]$ and $R_i \coloneqq (\mathcal{O}/(\pi))[\mathcal{G}_{k'_i}] = (\mathcal{O}/(\pi))[\mathcal{G}_{k_\infty}/U'_i]$. In particular, the residue field of R is also $\mathcal{O}/(\pi)$ so that $\overline{T} = \overline{\mathcal{T}}$ in this case. We therefore see that the validity of Hypothesis 4.14 (i) (iii) (iv) for T follows from Hypothesis 8.11. Similarly, Hypothesis 4.14 (ii*) follows from Hypothesis 8.11 (ii*). Furthermore, each R_i is a principal local ring because, as a power series ring over a field, R is a principal ideal domain. Given this, Lemma 8.20 applies to imply an inequality

$$\begin{aligned}\chi(\overline{F_i}, j(i)) \geq{}& \dim_{\mathbb{F}}(Y_{\Pi_k^\infty}(\overline{T})) + \dim_{\Bbbk}(\text{Ш}_{\overline{F_i}^*,j(i)}(\overline{T}^*(1))) - \dim_{\Bbbk}(\text{Ш}_{\overline{F_i},j(i)}(\overline{T}))\\ &+ \dim_{\Bbbk}\left(H^0(k,\overline{T})\right) - \dim_{\Bbbk}\left(H^0(k,\overline{T}^*(1))\right) - \sum\nolimits_{v\in\Sigma} H^0(k_v,\overline{T}^*(1)).\end{aligned}$$

Now, Hypothesis 8.11 (iii) implies the vanishing of $H^0(k,\overline{T}^*(1))$ via Remark 8.12. In addition, Lemma 6.1 (i) asserts that $\text{Ш}_{\overline{F_i}^*,j(i)}(\overline{T}^*(1)))$ is a submodule of $H^1(l_\infty(T)/k,\overline{T}^*(1))$ and so vanishes by Hypothesis 8.11 (iii). Similarly, $\text{Ш}_{\overline{F_i},j(i)}(\overline{T})$ is a submodule of $H^1(l_\infty(T)/k,\overline{T})$ by Lemma 6.1 (i). Given this, Hypothesis 8.11 (v) now ensures that $\chi(\overline{F_i}) > 0$, as required to verify Hypothesis 4.14 (vi).

We further take $\varrho\colon \mathcal{R} \to R$ to be the natural surjective projection map, and write $\mathfrak{P}$ for the prime ideal $\ker(\varrho)$ of $\mathcal{R}$. Then $\varrho_{\mathfrak{P}}$ is surjective and nonzero, as required by condition (i) in Theorem 4.20. To verify condition (ii) in Theorem 4.20, we note that since no finite place in $S(K)$ splits completely in k'_∞, the module $X_{S(K)}(\mathcal{T}) \otimes_{\Lambda_K} \mathcal{O}[\![\mathcal{G}_{k_\infty}]\!] \cong X_{S(K)}(T \otimes_{\mathcal{O}} \mathcal{O}[\![\mathcal{G}_{k_\infty}]\!])$ is finitely generated over $\mathcal{O}$. As a consequence of Lemma 8.22, this module is therefore annihilated by an element $x \in \mathcal{O}[\![\mathcal{G}_{k_\infty}]\!]$ that does not belong to the ideal $(\pi) = \ker(\mathcal{O}[\![\mathcal{G}_{k_\infty}]\!] \to R)$. It follows that $X_{S(K)}(\mathcal{T}) \otimes_{\Lambda_K} R$ is an R-torsion module and hence vanishes when localised at (0) as an R-module (respectively, at $\mathfrak{P}$ as an $\mathcal{R}$-module). This shows, in particular, that $\mathrm{Tor}_1^{\mathcal{R}}(X_{S(K)}(\mathcal{T}), R)_{\mathfrak{P}} = 0$ and so implies that

$$\mathrm{Fitt}^0_R(\mathrm{Tor}_1^{\mathcal{R}}(X_{S(K)}(\mathcal{T}), R))_{\mathfrak{P}} = \mathrm{Fitt}^0_{R_{\mathfrak{P}}}(\mathrm{Tor}_1^{\mathcal{R}}(X_{S(K)}(\mathcal{T}), R)_{\mathfrak{P}}) = \mathrm{Fitt}^0_{R_{\mathfrak{P}}}(0) = R_{\mathfrak{P}},$$

as required by condition (ii) in Theorem 4.20. In this case, therefore, the latter result can be applied in order to deduce that, for every Euler system c in $\mathrm{ES}^r_{\Sigma,S_0}(\mathcal{T})$, the element $(c_k)_{\mathfrak{P}}$ belongs to $\vartheta_Y(\mathrm{Det}_{\mathcal{R}}(C_{S(K),\Sigma}(\mathcal{T})))_{\mathfrak{P}}$. Since $\mathfrak{p}$ is contained in $\mathfrak{P}$, this in turn implies that $(c_k)_{\mathfrak{p}} \in \vartheta_Y(\mathrm{Det}_{\mathcal{R}}(C_{S(K),\Sigma}(\mathcal{T})))_{\mathfrak{p}}$ and hence verifies the $\mathfrak{p}$-localisation of (8.27).

(2) The case $p \notin \mathfrak{p}$

In this case, $\Lambda_{K,\mathfrak{p}}$ can be identified with the localisation $\Lambda_{\chi\psi,\wp}$ of $\Lambda_{\chi\psi}$ at a height-one prime $\wp \subseteq \Lambda_{\chi\psi}$ for some character $\psi\colon \Box_K \to \Phi^{c,\times}$. As a consequence, one also has an identification

$$\Theta_{K_\infty,\Sigma,b_\bullet}(\mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T})))_{\mathfrak{p}} = \Theta_{K_{\chi\psi,\infty},\Sigma,b_{\chi\psi,\bullet}}(\mathrm{Det}_{\Lambda_{\chi\psi}}(C_{S(K),\Sigma}(T(\chi\psi) \otimes_{\mathcal{O}_\psi} \Lambda_\psi)))_{\wp},$$

where $K_{\chi\psi,\infty} \coloneqq K_{\chi\psi} \cdot k_\infty$ with $K_{\chi\psi} \coloneqq K^{(\ker(\chi\psi))}$ is the field cut out by $\chi\psi$, we have denoted by $b_{\chi\psi,\bullet}$ the basis of $Y \otimes_{\Lambda_K} \Lambda_{\chi\psi}$ induced by $b_\bullet$, and $\mathcal{T}_{\chi\psi} \coloneqq T(\chi\psi) \otimes_{\mathcal{O}_{\chi\psi}} \Lambda_{\chi\psi}$.

On the other hand, using the twisting map $\mathrm{Tw}^r_{\chi\psi}\colon \mathrm{ES}^r_\Sigma(T) \to \mathrm{ES}^r_\Sigma(T(\chi\psi), \mathcal{K}^\Sigma)$ from Lemma 8.17 one has an equality $\mathrm{Tw}^r_{\chi\psi}(c)_{k_\infty} = \sum_{\sigma\in\Delta_K}(\chi\psi)(\sigma)\sigma c_{K_\infty}$ in

$$\big(\bigcap\nolimits^r_{\Lambda_K} H^1_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T})\big)_{\mathfrak{p}} = \big(\bigcap\nolimits^r_{\Lambda_\psi} H^1_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}_{\chi\psi})\big)_{\wp}$$

by [92, Lem. 4.3]. Now in $\Lambda_{K,\mathfrak{p}}$ the element $\sum_{\sigma\in\Delta_K}(\chi\psi)(\sigma)\sigma$ acts as multiplication by the unit $|\Delta_K|$ of $\Lambda_{K,\mathfrak{p}}$. Since $\mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T}))_{\mathfrak{p}}$ identifies with $\mathrm{Fitt}^r_{\Lambda_{\chi\psi}}(X_{S(K)}(\mathcal{T}_{\chi\psi}))_{\wp}$, the claimed result will follow if we can show that

$$\mathrm{Fitt}^r_{\Lambda_{\chi\psi}}(X_{S(K)}(\mathcal{T}_{\chi\psi})) \cdot \mathrm{Tw}^r_{\chi\psi}(c)_{k_\infty} \in \Theta_{K_{\chi\psi,\infty},\Sigma,b_{\chi\psi,\bullet}}(\mathrm{Det}_{\Lambda_{\chi\psi}}(C_{S(K),\Sigma}(\mathcal{T}_{\chi\psi})))_{\wp}.$$

By replacing c by $\mathrm{Tw}^r_{\chi\psi}(c)$ and T by $T(\chi\psi)$, we may thereby reduce to the case $K = k$ and $\mathcal{R} = \mathcal{O}[\mathrm{im}\,\chi][\![\Gamma]\!]$. Indeed, since ψ is a character of p-power order, one has that $\mathcal{O}_{\chi\psi}$ is a totally ramified extension of $\mathcal{O}_\chi$ with residue field equal to the residue field κ of $\mathcal{O}_\chi$. Writing $\pi_{\psi\chi}$ for a uniformiser of $\mathcal{O}_{\chi\psi}$, one has $\psi(\sigma) \equiv 1 \mod \pi_{\chi\psi}$ for every $\sigma \in G_k$ and so $\overline{T(\chi)} = \overline{T(\chi\psi)}$. This shows that the assumed validity of Hypotheses 8.11 for the representation $T(\chi)$ implies their validity also for $T(\chi\psi)$.

Now, since $\mathcal{R}$ is a factorial ring, the prime $\mathfrak{p}$ is principal and generated by a nonzero divisor $f \in \mathcal{R}\setminus(\pi)$, say. We can then extend (π, f) to a regular sequence $\sigma \coloneqq (\pi, f, x_1, \dots, x_{n-1})$ with $n \coloneqq \mathrm{rk}_{\mathbb{Z}_p}\Gamma$. We write I and J for the ideals of $\mathcal{R}$ generated by σ and $\sigma\setminus\{\pi\}$ respectively and fix a minimal prime $\mathfrak{P}$ containing J. For each natural number i we set $\mathfrak{a}_i \coloneqq I^i$ and $\mathcal{R}_i \coloneqq \mathcal{R}/\mathfrak{a}_i$. Then, since σ is an $\mathcal{R}$-regular sequence, each $\mathcal{R}_i$ is a zero-dimensional Gorenstein ring and $\mathfrak{P}$ has height n (by Krull's height theorem). In particular, since $\mathcal{R}/I = (\mathcal{O}/\pi) \otimes_{\mathcal{O}} (\mathcal{R}/J)$ is finite, the ring $\mathcal{R}/J$ is finitely generated over $\mathbb{Z}_p$ by Nakayama's lemma. The integral domain $\mathcal{R}/\mathfrak{P}$ is therefore a non-zero, finitely generated $\mathbb{Z}_p$-algebra that is $\mathbb{Z}_p$-free (since $\pi \notin \mathfrak{P}$) and so spans a finite field extension $F \cong \mathcal{R}_{\mathfrak{P}}/\mathfrak{P}\mathcal{R}_{\mathfrak{P}}$ of $\mathbb{Q}_p$. The integral closure R of $\mathcal{R}/\mathfrak{P}$ in F is then a discrete valuation domain and, for each i, we set $R_i \coloneqq R/\pi^i$. We write ϱ for the natural map $\mathcal{R} \to R$ and note that $\ker(\varrho) = \mathfrak{P}$ so that the localised map $\varrho_{\mathfrak{P}}\colon \mathcal{R}_{\mathfrak{P}} \to R_{\mathfrak{P}} = F$ is non-zero and surjective (since $p \notin \mathfrak{P}$), as required to verify condition (i) in Theorem 4.20. In addition, Lemma 8.20 applies because each R_i is a principal ideal ring and so, just as in case (i), Hypotheses 8.11 (iii) and (v) combine to imply $\chi(F_i) > 0$. Theorem 4.20 can therefore be applied in this context to deduce that, for all elements x and y of $\mathrm{Fitt}^r_{\mathcal{R}}(X_{S(K)}(\mathcal{T}))$, one has

$$xy^N \cdot (c_{K_\infty})_{\mathfrak{P}} \in y^N \cdot \vartheta_Y(\mathrm{Det}_{\mathcal{R}}(C_{S(K),\Sigma}(\mathcal{T})))_{\mathfrak{P}}.$$

Since $\mathfrak{p} \subseteq \mathfrak{P}$, this inclusion remains valid if one replaces $\mathfrak{P}$ by $\mathfrak{p}$. To cancel the term y^N from the resulting inclusion, and hence verify the $\mathfrak{p}$-localisation of (8.27), it is then enough to ensure that y is a nonzero divisor in Λ_K. In addition, the assumption that no finite place of k splits completely in k_∞ ensures that $X_{S(K)}(\mathcal{T}) = \bigoplus_{w\in S(K)_{K_\infty}} H^0(K_{\infty,w}, T^\vee(1))^\vee$ is finitely

generated over a power series ring in $d-1$ variables, and so is a Λ_K-torsion module. The ideal $\mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T}))$ therefore contains nonzero divisors and so it is enough to take y to be one of these elements.
This concludes the proof of Proposition 8.26. □

To prove Theorem 8.16 (i), it is now enough to show that the conclusion of Proposition 8.26 remains valid if one replaces the term $\mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T}))$ by $\mathrm{Fitt}^r_{\Lambda_K}(X_{S_0}(\mathcal{T}))$. To do this, we can argue locally at each $\mathfrak{p} \in \mathrm{Spec}^1(\Lambda_K)$.
If firstly $p \in \mathfrak{p}$, then one has $\mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T}))_{\mathfrak{p}} = \mathrm{Fitt}^r_{\Lambda_K}(X_{S_0}(\mathcal{T}))_{\mathfrak{p}} = \Lambda_{K,\mathfrak{p}}$. Indeed, the modules $X_{S(K)}(\mathcal{T})$ and $X_{S_0}(\mathcal{T})$ are both finitely generated over a power series ring in $d-1$ variables (as already observed above) and hence vanish when localised at $\mathfrak{p}$ as a consequence of Lemma 8.22. We can therefore assume $p \notin \mathfrak{p}$. In this case $\Lambda_{K,\mathfrak{p}}$ identifies with $\Lambda_{K_\psi,\mathfrak{p}_\psi}$ for some character $\psi\colon \Delta_K \to \Phi^{c,\times}$ and prime $\mathfrak{p}_\psi \in \mathrm{Spec}^1(\Lambda_{K_\psi})$ (with $p \notin \mathfrak{p}_\psi$) and the trace element $\mathrm{N}_{K_\infty/K_{\psi,\infty}}$ of $\mathcal{O}[\Delta_K]$ is a unit in $\Lambda_{K,\mathfrak{p}}$. We set $\mathcal{T}_\psi \coloneqq T(\psi) \otimes_{\mathcal{O}_\psi} \Lambda_\psi$.
Now, if v belongs to $S(K) \setminus S_0$, then Lemma 8.23 (i) asserts that $\mathrm{Fitt}^0_{\Lambda_{K_\psi}}(H^0(k_v, \mathcal{T}_\psi^\vee(1))^\vee)_{\mathfrak{p}_\psi}$ is equal to $\Lambda_{K_\psi,\mathfrak{p}_\psi}$ if v is in $S_0(K_\psi)$ (because $v \in S_{\mathrm{ram}}(K_\psi/k)$ in this case) and is generated by $\mathrm{Eul}_v(\mathrm{Frob}_v^{-1})$ otherwise. As a consequence, we have

$$\begin{aligned}\mathrm{Fitt}^r_{\Lambda_{K_\psi}}(X_{S(K)}(\mathcal{T}_\psi))_{\mathfrak{p}_\psi} &= \mathrm{Fitt}^r_{\Lambda_{K_\psi}}(X_{S_0}(\mathcal{T}_\psi))_{\mathfrak{p}_\psi} \cdot \mathrm{Fitt}^0_{\Lambda_{K_\psi}}(Y_{S(K)\setminus S_0}(\mathcal{T}_\psi))_{\mathfrak{p}_\psi} \\ &= \mathrm{Fitt}^r_{\Lambda_{K_\psi}}(X_{S_0}(\mathcal{T}_\psi))_{\mathfrak{p}_\psi} \cdot \big(\prod_{v \in S(K)\setminus S_0(K_\psi)} \mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\big), \end{aligned} \tag{8.28}$$

where the first equality follows as a consequence of the exact sequence

$$0 \longrightarrow X_{S_0}(\mathcal{T}_\psi) \longrightarrow X_{S(K)}(\mathcal{T}_\psi) \longrightarrow Y_{S(K)\setminus S_0}(\mathcal{T}_\psi) \longrightarrow 0.$$

The $\mathfrak{p}$-component of the inclusion in Theorem 8.16 (i) therefore follows from the computation

$$\begin{aligned}&\mathrm{Fitt}^r_{\Lambda_K}(X_{k,S_0}(\mathcal{T}))_{\mathfrak{p}} \cdot \big(\prod_{v \in S(K)\setminus S_0(K)} \mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\big) \cdot c_{K_\infty} \\ &= \mathrm{Fitt}^r_{\Lambda_{K_\psi}}(X_{S_0}(\mathcal{T}_\psi))_{\mathfrak{p}_\psi} \cdot \big(\prod_{v \in S(K)\setminus S_0(K)} \mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\big) \cdot (\mathrm{N}_{K_\infty/K_{\psi,\infty}} c_{K_\infty}) \\ &= \mathrm{Fitt}^r_{\Lambda_{K_\psi}}(X_{k,S_0}(\mathcal{T}_\psi))_{\mathfrak{p}_\psi} \cdot \big(\prod_{v \in S_0(K)\setminus S_0(K_\psi)} \mathrm{Eul}_v(\mathrm{Frob}_v^{-1})\big) \cdot c_{K_{\psi,\infty}} \\ &= \mathrm{Fitt}^r_{\Lambda_{K_\psi}}(X_{S(K)}(\mathcal{T}_\psi))_{\mathfrak{p}_\psi} \cdot c_{K_{\psi,\infty}} \\ &\subseteq \Theta_{K_{\psi,\infty},\Sigma,\boldsymbol{b}_\bullet}(\mathrm{Det}_{\Lambda_{K_\psi}}(C_{S(K),\Sigma}(\mathcal{T}_\psi)))_{\mathfrak{p}_\psi} \\ &= \Theta_{K_\infty,\Sigma,\boldsymbol{b}_\bullet}(\mathrm{Det}_{\Lambda_K}(C_{S(K),\Sigma}(\mathcal{T})))_{\mathfrak{p}}.\end{aligned}$$

Here the first equality holds because $\mathrm{N}_{K_\infty/K_{\psi,\infty}}$ is a unit in $\Lambda_{K,\mathfrak{p}}$, the second equality is by the Euler system norm relations, the third equality is by (8.28), and the inclusion follows from Proposition 8.26 with K replaced by K_ψ.
This concludes the proof of claim (i) of Theorem 8.16.
To prove (ii), we first make the general observation that for any subring R of $\mathcal{Q}_K$ that contains Λ_K, subsequent applications of Lemma 2.11 (iii) and (v) shows that

$$\begin{aligned}\mathrm{Fitt}^r_R(R \otimes_{\Lambda_K} Y_{\Pi_k^\infty}(\mathcal{T})) &= \mathrm{Fitt}^0_R\big(R \otimes_{\Lambda_K} \ker(Y_{\Pi_k^\infty}(\mathcal{T}) \to Y)\big) \\ &\subseteq \mathrm{Ann}_{\mathcal{Q}_K}\big(\mathcal{Q}_K \otimes_{\Lambda_K} \ker(Y_{\Pi_k^\infty}(\mathcal{T}) \to Y)\big) \\ &= \mathcal{Q}_K \epsilon_K \end{aligned} \tag{8.29}$$

with equality if $R = \mathcal{Q}_K$. Now, $X_{S(K)_{\mathrm{fin}}}(\mathcal{T})$ is a Λ_K-torsion module because no finite place of k splits completely in k_∞. It follows that

$$\mathcal{Q}_K \otimes_{\Lambda_K} \mathrm{Fitt}^0_{\Lambda_K}(X_{S(K)_{\mathrm{fin}}}(\mathcal{T})) = \mathrm{Fitt}^0_{\mathcal{Q}_K}(\mathcal{Q}_K \otimes_{\Lambda_K} X_{S(K)_{\mathrm{fin}}}(\mathcal{T})) = \mathrm{Fitt}^0_{\Lambda_K}(0) = \mathcal{Q}_K,$$

and this combines with the exact sequence $0 \to X_{S(K)_{\rm fin}}(\mathcal{T}) \to X_{S(K)}(\mathcal{T}) \to Y_{k,\Pi_k^\infty}(\mathcal{T}) \to 0$ and (8.29) to imply that

$$\mathcal{Q}_K \otimes_{\Lambda_K} \mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T})) = \mathcal{Q}_K \epsilon_K.$$

From Proposition 8.26 we therefore obtain

$$\epsilon_K(c_{K_\infty})_\chi \in \mathcal{Q}_K \cdot \Theta_{K_\infty,\Sigma,b_\bullet}(\mathrm{Det}_{\Lambda_K}(C_{S,\Sigma}(\mathcal{T}))) = (\mathcal{Q}_K e_K \epsilon_K) \otimes_{\Lambda_K} \bigcap\nolimits^r_{\Lambda_K} H^1_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}),$$

where the equality is by definition of the map $\Theta_{K_\infty,\Sigma,b_\bullet}$. This shows that e_K acts as the identity on $\epsilon_K(c_{K_\infty})_\chi$ and so, because e_K annihilates $\mathcal{Q}_K \otimes_{\Lambda_K} H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T})$ by definition of e_K, we see that $\mathcal{Q}_K \otimes_{\Lambda_K} H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T})_\chi$ can only be supported on primes $\mathfrak{p}$ of $\mathcal{Q}_K$ for which $\epsilon(c_{K_\infty})_\chi$ vanishes. Now, Proposition 3.45 (v) gives an exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee \longrightarrow H^2_\Sigma(\mathcal{O}_{k,S}, \mathcal{T}) \longrightarrow X'_{S\setminus\Pi_k^\infty}(\mathcal{T}) \longrightarrow 0. \tag{8.30}$$

which combines with the vanishing of $\mathcal{Q}_K \otimes_{\Lambda_K} X_{S(K)_{\rm fin}}(\mathcal{T})$ to imply that $\mathcal{Q}_K \otimes_{\Lambda_K} H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}) = \mathcal{Q}_K \otimes_{\Lambda_K} H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee$. This discussion therefore proves claim (ii) of Theorem 8.16.

To prove (iii), we note Proposition A.9 (ii) implies, for every $\mathfrak{p} \in \mathrm{Spec}^1(\Lambda_K)$, an equality

$$\{f(a) \mid a \in \mathrm{im}(\Theta_{K_\infty,\Sigma,b_\bullet}), f \in \bigwedge\nolimits^r_{\Lambda_K} H^1_\Sigma(\mathcal{O}_{k,S}, \mathcal{T})^*\}_\mathfrak{p} = \mathrm{Fitt}^r_{\Lambda_K}(H^1(C_{S,\Sigma}(\mathcal{T})))_\mathfrak{p}. \tag{8.31}$$

In addition, from Proposition 5.9 one has an exact sequence

$$0 \longrightarrow H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}) \longrightarrow H^1(C_{S(K),\Sigma}(\mathcal{T})) \longrightarrow Y_{\Pi_k^\infty}(\mathcal{T}) \longrightarrow 0, \tag{8.32}$$

which is split-exact if (as we are assuming) p satisfies condition (3.1).

Now, if $\mathfrak{p} \in \mathrm{Spec}^1(\Lambda_{K,\chi}) \subseteq \mathrm{Spec}^1(\Lambda_K)$ with $p \notin \mathfrak{p}$, then $\Lambda_{K,\mathfrak{p}}$ is a discrete valuation ring and so $\Lambda_{K,\mathfrak{p}}$-Fitting ideals are multiplicative on short exact sequences. Hence, in this case, for $f \in \bigwedge^r_{\Lambda_K} H^1_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T})^*$, one has that

$$\begin{aligned}
\mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T}))_\mathfrak{p} \cdot f(c_K)_\mathfrak{p} &\subseteq \mathrm{Fitt}^r_{\Lambda_K}(H^1(C_{S(K),\Sigma}(\mathcal{T})))_\mathfrak{p} \\
&= \mathrm{Fitt}^r_{\Lambda_K}(Y_{\Pi_k^\infty}(\mathcal{T}))_\mathfrak{p} \cdot \mathrm{Fitt}^0_{\Lambda_K}(H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}))_\mathfrak{p} \\
&\subseteq \epsilon_K \, \mathrm{Fitt}^0_{\Lambda_K}(H^2_\Sigma(\mathcal{O}_{k,S(K)}, \mathcal{T}))_\mathfrak{p} \\
&= \epsilon_K \, \mathrm{Fitt}^0_{\Lambda_K}(X_{S(K)\setminus\Pi_k^\infty}(\mathcal{T}))_\mathfrak{p} \cdot \mathrm{Fitt}^0_{\Lambda_K}(H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee)_\mathfrak{p} \\
&= \epsilon_K \, \mathrm{Fitt}^r_{\Lambda_K}(X_{S(K)}(\mathcal{T}))_\mathfrak{p} \cdot \mathrm{Fitt}^0_{\Lambda_K}(H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee)_\mathfrak{p}.
\end{aligned}$$

Here the first inclusion is by Proposition 8.26 and (8.31), the first equality is by the exact sequence (8.32), the second inclusion is by (8.29), the second equality by (8.30), and the final equality follows from (8.30). By cancellation, we can therefore deduce the required containment

$$\epsilon_K f(c_K)_\mathfrak{p} \in \epsilon_K \, \mathrm{Fitt}^0_{\Lambda_K}(H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee)_\mathfrak{p}. \tag{8.33}$$

We next prove the same result for $\mathfrak{p} \in \mathrm{Spec}^1(\Lambda_{K,\chi})$ with $p \in \mathfrak{p}$. For this, we note the assumption that no finite place splits completely in k_∞ combines with Lemma 8.22 to imply, for each such $\mathfrak{p}$, that $X_{S(K)\setminus\Pi_k^\infty}(\mathcal{T})_\mathfrak{p} = (0)$. In this case, it thus follows that $\mathrm{Fitt}^0_{\Lambda_K}(X_{S(K)_{\rm fin}}(\mathcal{T}))_\mathfrak{p} = \Lambda_{K,\mathfrak{p}}$, and the sequence (8.30) implies $H^1(C_{S(K),\Sigma}(\mathcal{T}))_\mathfrak{p}$ is isomorphic to the direct sum of $(H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee)_\mathfrak{p}$ and $Y_{\Pi_k^\infty}(\mathcal{T})$. Combining this with (8.29), we then see that the ideal $\mathrm{Fitt}^r_{\Lambda_K}(H^1(C_{S(K),\Sigma}(\mathcal{T})))_\mathfrak{p}$ is contained in $\epsilon_K H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee)_\mathfrak{p}$. For each such $\mathfrak{p}$, the required containment therefore follows directly from Proposition 8.26 and the equality (8.31).

At this stage, we have proved (8.33) for every $\mathfrak{p} \in \mathrm{Spec}^1(\Lambda_{K,\chi})$. From Lemma 2.4 (ii), we can therefore deduce that $\epsilon_K f(c_K)_\chi \in \epsilon_K \, \mathrm{Fitt}^0_{\Lambda_K}(H^1_{\mathcal{F}^*_{\mathrm{rel},\Sigma}}(k, \mathcal{T}^\vee(1))^\vee)^{**}$, as required to prove (iii).

This concludes the proof of Theorem 8.16. □

9. Elliptic curves

In this section we let E be an elliptic curve defined over $\mathbb{Q}$ and of conductor $N \coloneqq N_E$. We then consider the integral and rational p-adic Tate modules of E defined as

$$T_pE \coloneqq \varprojlim_{n\in\mathbb{N}} E[p^n] \quad \text{and} \quad V_pE \coloneqq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_pE.$$

We fix a $\mathbb{Z}_p$-basis of T_pE and hence identify $\mathrm{Aut}_{\mathbb{Z}_p}(T_pE)$ with $\mathrm{GL}_2(\mathbb{Z}_p)$ (the precise choice of this basis will not matter to our arguments). In this way, the natural action of $G_\mathbb{Q}$ on T_pE gives rise to a homomorphism $\rho_{E,p}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{Z}_p)$.
For any subfield $\mathcal{K}$ of $\mathbb{Q}^c$ we write $\Omega(\mathcal{K})$ for the set of finite abelian extensions of $\mathbb{Q}$ in $\mathcal{K}$. For K in $\Omega(\mathbb{Q}^c)$, we set $\mathcal{G}_K \coloneqq \mathrm{Gal}(K/\mathbb{Q})$ and $\widehat{\mathcal{G}_K} \coloneqq \mathrm{Hom}_\mathbb{Z}(\mathcal{G}_K, \mathbb{Q}^{c,\times})$.

9.1. Kato's Euler system of zeta elements

We review the Euler system for T_pE constructed by Kato in [60]. To do this, we fix a minimal Weierstrass model of E over $\mathbb{Z}$ and write $\omega \in H^0_{\mathrm{dR}}(E, \Omega^1_{E/\mathbb{Q}})$ for the corresponding Néron differential. Write $c_\infty \in \{1,2\}$ for the number of connected components of $E(\mathbb{R})$, and define periods of E by setting

$$\Omega^+ = \Omega^+_{\omega,\gamma} \coloneqq \int_{E(\mathbb{R})} \mid \omega \mid = c_\infty \cdot \int_{\gamma^+} \omega \quad \text{and} \quad \Omega^- = \Omega^-_{\omega,\gamma} \coloneqq \int_{\gamma^-} \omega,$$

where γ^+ and γ^- are generators of the subgroups $H_1(E(\mathbb{C}),\mathbb{Z})^+$ and $H_1(E(\mathbb{C}),\mathbb{Z})^-$ of $H_1(E(\mathbb{C}),\mathbb{Z})$ on which complex conjugation acts by $+1$ and -1, respectively, that are chosen in such a way that the real numbers Ω^+ and $(-i)\Omega^-$ are positive.
Fix an embedding $\iota\colon \mathbb{Q}^c \hookrightarrow \mathbb{C}$ and, given a number field K, write $w_\iota \coloneqq w_{\iota,K}$ for the place of K corresponding with the restriction ι_K of ι to K.
For each place $v \in \Pi^p_K$, Kato [57, Ch. II, § 1.2.4] defines a 'dual exponential map'

$$\exp^*_{K_v}\colon H^1(K_v, V_pE) \to \mathrm{Fil}^0 D_{\mathrm{dR},K_v}(V_pE). \tag{9.1}$$

We set $S_0 \coloneqq \{p, w_{\iota,\mathbb{Q}}\}$ and $S_1 \coloneqq S \cup \{\ell \mid N\}$. For each $K \in \Omega(\mathbb{Q}^c)$ we then also set $S_0(K) \coloneqq S_0 \cup S_{\mathrm{ram}}(K/\mathbb{Q})$ and $S(K) \coloneqq S_1 \cup S_{\mathrm{ram}}(K/\mathbb{Q})$.

(9.2) Theorem (Kato). *There exists a collection of elements*

$$z^{\mathrm{Kato}} \coloneqq (z_K^{\mathrm{Kato}})_K \in \prod_{K\in\Omega(\mathbb{Q}^c)} H^1(\mathcal{O}_{K,S(K)}, V_pE)$$

with the following properties.

(a) For all L and K in $\mathscr{F}(\mathbb{Q}^c)$ such that $K \subseteq L$, one has

$$\mathrm{Cores}_{L/K}(z_L^{\mathrm{Kato}}) = \big(\prod_{\ell\in S(L)\setminus S(K)} \mathrm{Eul}_\ell(\mathrm{Frob}_\ell^{-1})\big) \cdot z_K^{\mathrm{Kato}},$$

where $\mathrm{Cores}_{L/K}\colon H^1(L, V_pE) \to H^1(K, V_pE)$ denotes the corestriction map.

(b) Set

$$y_K^{\mathrm{Kato}} \coloneqq \big(\prod_{\ell\in S(K)\setminus S_0(K)} \mathrm{Eul}_\ell(\mathrm{Frob}_\ell^{-1})\big)^{-1} z_K^{\mathrm{Kato}}.$$

If $E[p]$ is irreducible as an $\mathbb{F}_p[G_\mathbb{Q}]$-module, then $c_\infty y_K^{\mathrm{Kato}}$ (and hence also $c_\infty z_K^{\mathrm{Kato}}$) belongs to $H^1(\mathcal{O}_{K,S(K)}, T_pE)$. In particular, the collections z^{Kato} and $y^{\mathrm{Kato}} \coloneqq (y_K^{\mathrm{Kato}})_{K\in\mathscr{F}(\mathbb{Q}^c)}$ respctively belong to $\mathrm{ES}^1_{\varnothing,S}(T_pE)$ and $\mathrm{ES}^1_{\varnothing,S_1}(T_pE)$.

(c) For every $K \in \mathscr{F}(\mathbb{Q}^c)$, one has an equality

$$(\oplus_{v\in\Pi^p_K} \exp^*_{K_v})(z_K^{\mathrm{Kato}}) = \big(\sum_{\chi\in\widehat{G_K}} \frac{L_{S(K)}(E,\chi^{-1},1)}{\Omega^{\mathrm{sgn}(\chi)}} e_\chi\big) \otimes \omega_E$$

in $\bigoplus_{v\in\Pi^p_K} \mathrm{Fil}^0_{\mathrm{dR},K_v}(V_pE) \cong \mathbb{Q}_p \otimes_\mathbb{Q} H^0_{\mathrm{dR}}(E, \Omega^1_{E/K}) = (\mathbb{Q}_p \otimes_\mathbb{Q} K) \otimes_\mathbb{Q} H^0_{\mathrm{dR}}(E, \Omega^1_{E/\mathbb{Q}})$.

Proof. The elements z_K^{Kato} and y_K^{Kato} are defined by slightly modifying the elements constructed by Kato in [60, (8.1.3)]. The integrality property in claim (b) is proved by the argument of [60, § 12.6], and the 'explicit reciprocity law' in claim (c) is a consequence of [60, Th. 9.7 and 12.5]. If p is an odd prime of good reduction, then the details for this argument are given by Kataoka in [54, Th. 6.1], and this extends to any prime p for which $E[p]$ is irreducible as in [25, § 6.3]. (One only has to replace the application of [114, Prop. 8] by the representation-theoretic observation in [25, Rk. 6.9].) □

(9.3) Remark. (a) Regarding Theorem 9.2 (b), it can be shown that, if $\mathrm{SL}_2(\mathbb{Z}_p) \subseteq \mathrm{im}(\rho_{E,p})$, then $E[p]$ is an irreducible $\mathbb{F}_p[G_\mathbb{Q}]$-module and also $E(K)[p] = (0)$ for every $K \in \Omega(\mathbb{Q}^c)$.

(b) An analogue of the integrality property in Theorem 9.2 (b) also holds when $E[p]$ is a reducible $\mathbb{F}_p[G_\mathbb{Q}]$-representation. However, such an extension of Theorem 9.2 requires a detailed discussion of Kato's argument in [60, § 12.6] and so, since it is outside the scope of Euler system theory developed in the present article, this will be discussed elsewhere.

9.2. Kato's Iwasawa main conjecture

Given a finite abelian extension K of $\mathbb{Q}$, we write $K_\infty \coloneqq \bigcup_{n\in\mathbb{N}} K_n$ for its cyclotomic $\mathbb{Z}_p$-extension. We let $\Lambda_K \coloneqq \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/\mathbb{Q})]\!]$ denote the associated Iwasawa algebra, and $\mathcal{Q}_K$ its total ring of fractions.

Define the representation $\mathcal{T} \coloneqq (T_pE)\otimes_{\mathbb{Z}_p}\Lambda_K$ and note that $H^1(\mathcal{O}_{\mathbb{Q},S(K)}, \mathcal{T})$ equals the Iwasawa cohomology group $H^1_{\mathrm{Iw}}(\mathcal{O}_{K,S(K)}, T_pE) \coloneqq \varprojlim_n H^1(\mathcal{O}_{K_n,S(K)}, T_pE)$ (with the limit taken with respect to corestriction maps). The family $z_{K_\infty}^{\mathrm{Kato}} \coloneqq (z_{K_n}^{\mathrm{Kato}})_{n\in\mathbb{N}}$ therefore defines an element of $H^1_{\mathrm{Iw}}(\mathcal{O}_{K,S(K)}, T_pE)$.

Denote by $\tau \in G_\mathbb{Q}$ the unique element with the property that $\iota \circ \tau$ equals complex conjugation, and write $\tau_K \in \mathcal{G}_K$ for its restriction to K. For every $n \in \mathbb{N}$, we may then define a $\mathbb{Z}[1/2][\mathcal{G}_{K_n}]$-basis of

$$H_1((E \times \mathrm{Spec}\, K_n)(\mathbb{C}), \mathbb{Z}[1/2])^+ \cong \big(\mathbb{Z}[\mathcal{G}_{K_n}] \otimes_\mathbb{Z} H_1(E^\iota(\mathbb{C}), \mathbb{Z}[1/2])\big)^+$$

by means of

$$\gamma_{K_n} \coloneqq \big((1 + \tau_{K_n}) \otimes \gamma^+ + (1 - \tau_{K_n}) \otimes \gamma^-\big)/2.$$

For an odd prime p, we will regard γ_{K_n} as a basis of $Y_{K_n}(T_pE)$ via the comparison isomorphism $\mathbb{Z}_p \otimes_\mathbb{Z} H_1((E \times \mathrm{Spec}\, K_n)(\mathbb{C}), \mathbb{Z})^+ \cong Y_{K_n}(T_pE)$. Taking the limit over n, we obtain a Λ_K-basis γ_{K_∞} of $Y_{\Pi_k^\infty}(\mathcal{T})$. We will then use the map

$$\Theta_{K_\infty,S(K)} \coloneqq \Theta_{K_\infty,S(K),\gamma_{K_\infty}} \colon \mathrm{Det}_{\Lambda_K}(C_{S(K)}(\mathcal{T})) \to \mathcal{Q}_K \otimes_{\Lambda_K} H^1_{\mathrm{Iw}}(\mathcal{O}_{K,S(K)}, T_pE).$$

that was defined in (8.14).

(9.4) Theorem. *Fix $p > 3$ with $\mathrm{SL}_2(\mathbb{Z}_p) \subseteq \mathrm{im}(\rho_{E,p})$ and, if E does not have potentially good reduction at p, also $\mu_K[p] = (0)$. Then one has $z_{K_\infty}^{\mathrm{Kato}} \in \Theta_{K_\infty,S(K)}(C_{S(K)}(\mathcal{T}))$.*

(9.5) Remark. If E does not have CM, then Serre has proved in [100] that $\rho_{E,p}$ is surjective for all but finitely many prime numbers p, and asked if in fact $\rho_{E,p}$ is surjective when $p > 37$. It is conjectured that surjectivity is implied by $p \notin \{2, 3, 5, 11, 13, 17, 37\}$ and the following is known in this direction.

- Zywina has proved in [115, Th. 1.10] that a prime that fails surjectivity is bounded from above by $\max\{37, N_E\}$.
- If E is a semi-stable elliptic curve, then $\rho_{E,p}$ is surjective if $p \geq 11$ by a result of Mazur [73, Th. 4].
- Zywina has proved in [115, Th. 1.5] that if $\rho_{E,p}$ is not surjective (with $p > 13$) and $\ell \neq p$ is a prime at which E does not have potentially good reduction, then $\ell \equiv \pm 1 \mod p$ and p divides the Tamagawa number Tam_ℓ at ℓ.

Theorem 9.4 will be proved in § 9.3. However, we first record the following consequence that involves the induced representation $T_{K/\mathbb{Q}} := T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathcal{G}_K]$ and the map

$$\Theta_{K,S(K)} := \Theta_{K,S(K),\gamma_K} \colon \operatorname{Det}_{\mathbb{Z}_p[\mathcal{G}_K]}(C_{S(K)}(T_{K/\mathbb{Q}})) \to H^1(\mathcal{O}_{K,S(K)}, V_pE)$$

defined via (8.6).

(9.6) Corollary. *Fix $p > 3$ with $\mathrm{SL}_2(\mathbb{Z}_p) \subseteq \operatorname{im}(\rho_{E,p})$ and, if E does not have potentially good reduction at p, also $\mu_K[p] = (0)$.*

(i) z_K^{Kato} is contained in the image of $\Theta_{K,S(K)}$.

(ii) Assume the p-part of the Birch–Swinnerton-Dyer Conjecture holds for all subfields F of K with $p \nmid [F : \mathbb{Q}]$. Then z_K^{Kato} generates the image of $\Theta_{K,S(K)}$ over $\mathbb{Z}_p[\mathcal{G}_K]$.

Proof. At the outset we note that K_n contains a p-th root of unity if and only if K does. Given this, the first claim is immediate from Theorem 9.4 and the fact that $\Theta_{K_\infty,S(K),\gamma_K}$ and $z_{K_\infty}^{\mathrm{Kato}}$ are the limits (over n) of $\Theta_{K_n,S(K),\gamma_{K_n}}$ and $z_{K_n}^{\mathrm{Kato}}$, respectively.
To prove the second claim, we write $e_{0,K_n} := \sum_\chi e_\chi$ with the sum over all characters $\chi\colon \mathcal{G}_n \to \mathbb{C}^\times$ with $L(E,\chi,1) \neq 0$. A well-known application of Nakayama's lemma via the argument of [10, Prop. 3.6] (with the Birch–Swinnerton-Dyer conjecture playing the role of the analytic class number formula in loc. cit.) then shows that $e_{0,K_n}\mathbb{Z}_p[\mathcal{G}_n]z_{K_n}^{\mathrm{Kato}} = e_{0,K_n} \operatorname{im}\Theta_{K_n,S(K),\gamma_{K_n}}$. Set $\mathcal{G}_n := \mathcal{G}_{K_n}$ and suppose $a_0 := \Theta_{K,S,\gamma_K}(\alpha_0)$ does not belong to the image of $\mathbb{Z}_p[G]z_K^{\mathrm{Kato}}$. By lifting α_0 to an element $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of $\operatorname{Det}_{\Lambda_K}(C^\bullet_{S(K)}(\mathcal{T}))$, we may then regard a_0 as the bottom class of the family $a := (a_n)_{n\in\mathbb{N}} = (\Theta_{K_n,S(K),\gamma_{K_n}}(\alpha_n))_{n\in\mathbb{N}}$. Moreover, the discussion above shows that e_{0,K_n} annihilates the class $[a_n]$ of a_n in the quotient $Z_n := (\operatorname{im}\Theta_{K_n,S(K),\gamma_{K_n}})/(\mathbb{Z}_p[\mathcal{G}_n]z_{K_n}^{\mathrm{Kato}})$ for all $n \in \mathbb{N}$. Now, a famous result of Rohrlich [90, Th. 1] that $L(E,\chi,1) = 0$ only for finitely many characters χ that are unramified outside $S(K)$. In particular, any character $\chi\colon \mathrm{Gal}(K_\infty/\mathbb{Q}) \to \mathbb{C}^\times$ with $e_\chi e_{K_n,0} = 0$ must factor through a finite extension F of $\mathbb{Q}$. Hence, for every $\sigma \in \mathrm{Gal}(K_\infty/F)$, one has

$$(\sigma - 1)\cdot[a_n] = (\sigma-1)(1-e_{0,K_n})\cdot[a_n] = 0\cdot[a_n],$$

which shows that $[a_n]$ is fixed by σ. We have therefore proved that $([a_n])_{n\in\mathbb{N}}$ belongs to $(\varprojlim_{n\in\mathbb{N}} Z_n)^{\mathrm{Gal}(K_\infty/F)}$ and so in order to conclude that $[a_0]$ is trivial it is enough to prove that $\varprojlim_{n\in\mathbb{N}} Z_n = (\operatorname{im}\Theta_{K_\infty,S(K)})/(\Lambda_K z_{K_\infty}^{\mathrm{Kato}})$ has no non-trivial $\mathrm{Gal}(K_\infty/F)$-invariant elements. Since $\mathrm{Gal}(K_\infty/F)$ is an open subgroup of $\mathrm{Gal}(K_\infty/\mathbb{Q})$, this will follow (see, for example, [82, Prop. 5.3.19 (i)]) if we can show that $\operatorname{im}\Theta_{K_\infty,S(K)}$ and $\Lambda_K z_{K_\infty}^{\mathrm{Kato}}$ are both Λ_K-free modules. To justify this, we first note that $z_{K_n}^{\mathrm{Kato}}$ and $\operatorname{im}\Theta_{K_n,S(K),\gamma_{K_n}}$ are cyclic Λ_K-modules and that their annihilators are contained in $\mathbb{Z}_p[\mathcal{G}_n][e_{0,K_n}]$ for every $n \in \mathbb{N}$. Indeed, this follows directly from Kato's explicit reciprocity law in Theorem 9.2 (c) and the definition of $\Theta_{K_n,S(K),\gamma_{K_n}}$ upon noting that $e_{K_n}\cdot e_{0,K_n} = e_{0,K_n}$ (cf. [25, Lem. 6.1 (iii)]), respectively. It therefore suffices to prove that $\varprojlim_{n\in\mathbb{N}}\mathbb{Z}_p[\mathcal{G}_n][e_{0,K_n}]$ vanishes. Now, as above it follows Rohrlich's result [90] that $\varprojlim_{n\in\mathbb{N}}\mathbb{Z}_p[\mathcal{G}_n][e_{0,K_n}]$ is a submodule of Λ_K that is fixed by $\mathrm{Gal}(K_\infty/F)$. However, $\mathrm{Gal}(K_\infty/F)$ is an open subgroup of $\mathrm{Gal}(K_\infty/\mathbb{Q})$ and so Λ_K has no non-trivial elements that are fixed by it. This shows that $\varprojlim_{n\in\mathbb{N}}\mathbb{Z}_p[\mathcal{G}_n][e_{0,K_n}]$ vanishes, as required. □

(9.7) Remark. The result of Corollary 9.6 is related to $\mathrm{TNC}(h^1(E/K)(1), \mathbb{Z}_p[G])$. Indeed, the 'analytic-rank-0 component' of the latter conjecture asserts that $e_{0,K}\mathbb{Z}_p[G]\cdot z_K^{\mathrm{Kato}} = e_{0,K}\operatorname{im}\Theta_{K,S}$ and, assuming a natural generalisation of a conjecture of Perrin-Riou, the 'analytic-rank-at-most-one component' is equivalent to the statement that z_K^{Kato} generates the image of $\Theta_{K,S(K)}$ (see [25, § 6.2] for more details).

(9.8) Remark. Due to important work of many authors including Coates–Wiles, Kolyvagin, Rubin, Kato, Skinner–Urban, and others, there is by now a range of results regarding the validity of the Birch–Swinnerton-Dyer conjecture. We restrict ourselves here to only stating the recent main result of Burungale–Castella–Skinner in [29, Cor. 1.3.1]: The p-part of the Birch–Swinnerton-Dyer conjecture for $E/\mathbb{Q}$ holds if $p > 3$ is a prime number at which E has good ordinary reduction (so $p \nmid a_p$), the image of $\rho_{E,p}$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$, and $\mathrm{ord}_{s=1} L(E, s) \leq 1$.

Corollary 9.6 also has the following consequence towards the conjecture of Birch and Swinnerton-Dyer in analytic rank zero for abelian number fields.

(9.9) Corollary. *Assume $p > 3$ is a prime number as in Corollary 9.6. If $L(E/K, 1) \neq 0$, then one divisibility in the 'p-part' of the Birch–Swinnerton-Dyer Conjecture holds for E/K. That is, $Ш_{E/K}[p^\infty]$ is finite and one has*

$$\mathrm{ord}_p\Big(\frac{L(E/K,1)}{|d_K|^{-1/2}\cdot\Omega_K}\Big) \geq \mathrm{ord}_p\big(|Ш_{E/K}[p^\infty]|\cdot \mathrm{Tam}_{E/K}\big),$$

where d_K denotes the discriminant of K, $\Omega_K \coloneqq (\Omega^+)^{r_1+r_2}(\Omega^-)^{r_2}$ with r_1 and r_2 the number of real and complex places of K, and $\mathrm{Tam}_{E/K}$ *the product of Tamagawa numbers of E over K. In addition, equality holds in the above display if the Birch–Swinnerton-Dyer Conjecture for E holds for all subfields F of K with $p \nmid [F : \mathbb{Q}]$.*

Proof. Corollary 9.6 verifies one inclusion in the 'analytic-rank-zero component' of the conjecture $\mathrm{TNC}(h^1(E/K)(1), \mathbb{Z}_p[G])$. The claimed result therefore follows from the well-known functoriality properties of Kato's conjecture and the fact that $\mathrm{TNC}(h^1(E/K)(1), \mathbb{Z}_p)$ is equivalent to the p-part of the Birch–Swinnerton-Dyer conjecture (cf. [62, 108, 30]). For the convenience of the reader, we provide details for how to deduce the claim from results in the literature.
The assumption $L(E/K, 1) \neq 0$ implies that $e_{0,K} = 1$ and so Corollary 9.6 implies that there is a unique element $a \in \mathrm{Det}_{\mathbb{Z}_p[\mathcal{G}_K]}(C_{S(K)}(T_{K/\mathbb{Q}}))$ with $\Theta_{K,S(K)}(a) = z_K^{\mathrm{Kato}}$ and a is a $\mathbb{Z}_p[\mathcal{G}_K]$-basis if the the Birch–Swinnerton-Dyer conjecture for E holds for all subfields F of K with $p \nmid [F : \mathbb{Q}]$. In addition, the assumption also implies that $H^2(\mathcal{O}_{K,S(K)}, V_pE)$ vanishes and hence we obtain a composite isomorphism

$$\begin{aligned} j_p\colon \mathrm{Det}_{\mathbb{Q}_p[\mathcal{G}_K]}(C_{S(K)}(T_{K/\mathbb{Q}})) &\xrightarrow[\simeq]{\Theta_{K,S(K)}} H^1(\mathcal{O}_{K,S(K)}, V_pE) \xrightarrow[\simeq]{} \textstyle\bigoplus_{v\mid p} H^1(K_v, V_pE) \\ &\xrightarrow[\simeq]{\oplus_{v\mid p}\exp^*_{K_v}} \mathbb{Q}_p \otimes_{\mathbb{Q}} H^0(E, \Omega^1_{E/K}) \xrightarrow{\omega_E \mapsto 1} \mathbb{Q}_p \otimes_{\mathbb{Q}} K. \end{aligned}$$

From Kato's explicit reciprocity law in Theorem 9.2 (c) it then follows that we have

$$j_p(a) = \sum\nolimits_{\chi\in\widehat{G}} \frac{L_{S(K)}(E, \chi^{-1}, 1)}{\Omega^{\epsilon(\chi)}} e_\chi.$$

Now, 'forgetting' the $\mathbb{Z}_p[\mathcal{G}_K]$-module structure, j_p also induces an isomorphism

$$\widetilde{j}_p\colon \mathrm{Det}_{\mathbb{Z}_p}(C_{S(K)}(T_{K/\mathbb{Q}})) \xrightarrow{\simeq} \mathbb{Q}_p.$$

We also have a forgetful map $\mathrm{Det}_{\mathbb{Z}_p[\mathcal{G}_K]}(C_{S(K)}(T_{K/\mathbb{Q}})) \to \mathrm{Det}_{\mathbb{Z}_p}(C_{S(K)}(T_{K/\mathbb{Q}}))$ (see, for example, [10, Lem. 3.7 (b)]), and if we write $\tilde{a}$ for the image of a under this map, then [6, Ch. III, § 9.6, Prop. 3] (see also [10, Lem. 3.7 (b)]) implies that

$$\widetilde{j}_p(\tilde{a}) = \mathrm{N}_{(\mathbb{Q}_p\otimes_{\mathbb{Q}}K)/\mathbb{Q}_p}(j_p(a)) = \prod\nolimits_{\chi\in\widehat{\mathcal{G}_K}} \frac{L_{S(K)}(E, \chi^{-1}, 1)}{\Omega^{\epsilon(\chi)}} = \frac{L_{S(K)}(E/K, 1)}{\Omega_K}.$$

The claim then follows by combining this with the equality

$$\widetilde{j}_p(\mathrm{Det}_{\mathbb{Z}_p}(C_{S(K)}(T_{K/\mathbb{Q}})) = |d_K|^{-1/2}\cdot|Ш[p^\infty]|\cdot \mathrm{Tam}_{E/K}\cdot\big(\prod\nolimits_{v\in S(K)_{\mathrm{fin}}} \mathrm{Eul}_v(1)\big)\cdot\mathbb{Z}_p$$

that follows from [30, Prop. 2.1]. □

9.3. The proof of Theorem 9.4

We begin by establishing a useful preliminary result.

(9.10) Lemma. *$\bigoplus_{v|p}(E(K_{\infty,v})_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p)^\vee$ is a finitely generated $\mathbb Z_p$-module that is finite if E has potentially good reduction at p. If it is infinite, then each $(E(K_{\infty,v})_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p)^\vee$ has $\mathbb Z_p$-rank one and the induced character*

$$\psi\colon G_{\mathbb Q_p}\to \mathrm{Aut}((E(K_{\infty,v})_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p)^\vee_{\rm tf})\cong \mathbb Z_p^\times$$

is equal to $\omega\cdot\chi_{\rm cyc}^{-1}$ with a character $\omega\colon G_{\mathbb Q_p}\to\mathbb Z_p^\times$ of order at most two.

Proof. At the outset we recall (from § 8.3.4) that, for any $\mathfrak p\in \mathrm{Spec}^1_{p'}(\Lambda_K)$, there exists a character $\chi=\chi_{\mathfrak p}\colon \Delta_K\to\overline{\mathbb Q}^\times$ such that $\Lambda_{K,\mathfrak p}$ coincides with the localisation of $\Lambda_\chi\coloneqq \mathbb Z_p[\mathrm{im}\,\chi][\![\Gamma_K]\!]$ at a suitable prime $\varrho_\chi\in\mathrm{Spec}^1(\Lambda_\chi)$. We set $\Delta_\chi\coloneqq\ker\chi$ and $K_\chi\coloneqq K^{\Delta_\chi}$ and identify Λ_χ with $\mathbb Z_p[\mathrm{im}\,\chi][\![\mathrm{Gal}(K_{\chi,\infty}/K_\chi)]\!]$.
To justify the first claim, we note the obvious injective homomorphism

$$E(K_{\infty,v})_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p\hookrightarrow E(\mathbb Q_p^{\rm c})_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p\cong (T_pE)^\vee(1)$$

induces a surjective map

$$\bigoplus\nolimits_{v|\ell}T_pE(-1)\twoheadrightarrow\bigoplus\nolimits_{v|\ell}(E(K_{\infty,v})_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p)^\vee. \qquad (9.11)$$

This shows that the right hand module in (9.11) is a finitely generated $\mathbb Z_p$-module (because p is finitely decomposed in K_∞).
The finiteness claim, in turn, is a classical result of Imai [50] (cf. also Remark 8.25). To prove the remaining claim, we may assume that E does not have potentially good reduction at p. In this case there then exists a quadratic extension L of $\mathbb Q_p$ such that E has split-multiplicative reduction at p (see [103, Th. 5.3]). Fix a p-adic place v of K_∞ and set $F_\infty\coloneqq L(\mu_p)K_\infty$, the (local) cyclotomic $\mathbb Z_p$-extension of the composite F of $L(\mu_p)$ with the completion K_v of K at the restriction of v to K. Then $E(F_\infty)$ contains $E(K_{\infty,v})$ and so $(E(K_{\infty,v})_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p)^\vee$ is a quotient of $(E(F_\infty)_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p)^\vee$.
Recall that Tate's uniformisation theorem [103, Ch. V, Cor. 5.4] gives a $G_L\coloneqq\mathrm{Gal}(\mathbb Q_p^{\rm c}/L)$-equivariant isomorphism $E(\mathbb Q_p^{\rm c})\cong\mathbb Q_p^{{\rm c},\times}/q_E^{\mathbb Z}$ with the Tate period $q_E\in\mathcal O_L\setminus\mathcal O_L^\times$ of E. Taking $G_{F_\infty}\coloneqq\mathrm{Gal}(\mathbb Q_p^{\rm c}/F_\infty)$-invariants, we obtain an isomorphism

$$E(F_\infty)_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p\cong\{\zeta u+q_E^{\mathbb Z}\mid \zeta\in\mu_{p^\infty},u^{p^s}=q_E \text{ for some } s\in\mathbb N\}^{G_{F_\infty}}.$$

Suppose ζu represents a G_{F_∞}-invariant class in the right hand set. Since F_∞ contains μ_{p^∞}, it then follows that $(\sigma-1)u=(\sigma-1)(\zeta u)$ belongs to $q_E^{\mathbb Z}$. On the other hand, we know that $(\sigma-1)u\in\mu_{p^\infty}$ because u is a root of $X^{p^s}-q_E$ for some $s\in\mathbb N$, and so $(\sigma-1)u=1$ as $q_E^{\mathbb Z}\cap\mu_{p^\infty}=\{1\}$. This shows that u belongs to F_∞.
We next prove that there is a natural number a such that q_E is not a p^s-th power in $F_\infty^\times$ for any $s>a$. In combination with the discussion above, this then shows that the cokernel of the map $(\mathbb Q_p/\mathbb Z_p)(1)\hookrightarrow E(F_\infty)_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p$ is finite, from which it follows by taking duals that the torsion-free quotient of $(E(F_\infty)_{\rm tor}\otimes_{\mathbb Z}\mathbb Z_p)^\vee$ is isomorphic to $\mathbb Z_p(-1)$. This isomorphism is G_L-equivariant and so $\psi\chi_{\rm cyc}$ must factor through the group $\mathrm{Gal}(L/\mathbb Q_p)$ of order two. From this we deduce that ψ is of the claimed form $\omega\chi_{\rm cyc}^{-1}$.
To prove the remaining claim, suppose that $q_E=u^{p^s}$ for some $s\in\mathbb N$ and $u\in F_\infty$. Then u is a root of $X^{p^s}-q_E$ and so $[F(u):F]\le p^s$. We may therefore assume that u belongs to $F(\mu_{p^s})$. In particular, q_E belongs to the kernel of the restriction map

$$F^\times/(F^\times)^{p^s}=H^1(G_F,\mu_{p^s})\to H^1(G_{F(\mu_{p^s})},\mu_{p^s})=F(\mu_{p^s})^\times/(F(\mu_{p^s})^\times)^{p^s}.$$

By the inflation-restriction sequence, the kernel of this map identifies with the cohomology group $H^1(\mathrm{Gal}(F(\mu_{p^s})/F),\mu_{p^s})$ and hence vanishes because p is odd (see, for example, [81, Satz (4.8)]). This shows that q_E is a p^s-th power in F, and hence that s is bounded, as required to prove the claim. □

Turning now to the proof of Theorem 9.4, we first verify that Hypotheses 8.11 are satisfied for $(T,k_\infty,1)$ with $T\coloneqq T_pE(\chi)$ for a character $\chi\colon G_{\mathbb Q}\to\mathbb C^\times$ of order prime to p.

Note that parts (ii*) and (iv) of Hypotheses 8.11 are vacuous because we are assuming that $p > 3$. To proceed, we note that the existence of the Weil pairing implies that the full preimage of $\mathrm{SL}_2(\mathbb{Z}_p)$ under $\rho_{E,\mathfrak{p}}$ is equal to $G_{\mathbb{Q}(\mu_{p^\infty})}$. Since we are assuming that $\mathrm{SL}_2(\mathbb{Z}_p) \subseteq \mathrm{im}(\rho_{E,p})$, it follows that the map $\rho_{E,p}(G_{\mathbb{Q}(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p)$ is surjective. Now, $\mathrm{SL}_2(\mathbb{Z}_p)$ is a perfect group (because $p > 3$) and so also $\rho_{E,p}(G_{K(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p)$. Since the natural action of $\mathrm{SL}_2(\mathbb{F}_p)$ on $\mathbb{F}_p^{\oplus}$ is irreducible, this implies that $\overline{T}$ is an irreducible $\kappa[G_{\mathbb{Q}}]$-representation, as required by Hypothesis 8.11 (i). In addition, Hypothesis 8.11 (ii) is satisfied with $\tau \in G_{K(\mu_{p^\infty})}$ taken to be a preimage of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Moreover, Hypothesis 8.11 (iii) follows from $H^1(\mathrm{SL}_2(\mathbb{Z}_p), \mathbb{F}_p^{\oplus 2}) = (0)$ as in [25, Lem. 6.17 (ii)]. Finally, since also $H^0(k, \overline{T})$ vanishes and we are taking $\Sigma = \varnothing$, part (v) is satisfied because $r = 1$ and $H^1(l_\infty(T)/k, \overline{T}) = (0)$ (as follows from [25, Lem. 6.17]).
We may therefore now apply Theorem 8.16 (i) to the Euler system y^{Kato} from Theorem 9.2 to deduce (because $z^{\mathrm{Kato}}_{K_\infty} = (\prod_{\ell \in S(K)\setminus S_0(K)} \mathrm{Eul}_\ell(\mathrm{Frob}_\ell^{-1}))y^{\mathrm{Kato}}_{K_\infty}$) that

$$\begin{aligned}&\mathrm{Fitt}^0_{\Lambda_K}(\bigoplus\nolimits_{v|p}(E(K_{\infty,v})_{\mathrm{tor}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\vee))^{**} \cdot z^{\mathrm{Kato}}_{K_\infty}\\ &= \mathrm{Fitt}^0_{\Lambda_K}(\bigoplus\nolimits_{v|p}(E(K_{\infty,v})_{\mathrm{tor}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\vee))^{**} \cdot \big(\prod_{\ell \in S(K)\setminus S_0(K)} \mathrm{Eul}_\ell(\mathrm{Frob}_\ell^{-1})\big) \cdot y_{K_\infty}\\ &\in \Theta_{K_\infty,S(K)}(C_{S(K)}(\mathcal{T})).\end{aligned}$$

In addition, Lemma 9.10 shows that if $M := (E(K_{v,\infty})_{\mathrm{tor}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\vee$ is not finite for some $v \mid p$, then E does not have potentially good reduction at p, M has $\mathbb{Z}_p$-rank one, and $G_{\mathbb{Q}_p}$ acts on the torsion-free quotient M_{tf} via a character of the form $\omega\chi_{\mathrm{cyc}}$ with ω of order at most two. By assumption, K does not contain a primitive p-th root of unity in this case, and so the same is true for $K_{n,v}$ for every n. It follows that there is a subgroup of $\mathrm{Gal}(K_{\infty,v}/\mathbb{Q}_p)$ of index at most two that acts trivially on M_{tf}. However, E has only finite many torsion points over any finite extension of $\mathbb{Q}_p$ and so this shows that M_{tf} must be trivial. We have thereby proved that M is finite so that $\mathrm{Fitt}^0_{\Lambda_K}(M)^{**}$, which is uniquely determined by its localisations at height-one primes by Lemma 2.4 (ii), is equal to Λ_K.
This concludes the proof of Theorem 9.4. □

10. Tate motives

In the sequel, for a number field F and integer a we write $\mathbb{Q}_F(a)$ for the motive $h^0(\mathrm{Spec}\, F)(a)$. In particular, for a finite abelian extension K of k we regard each $\mathbb{Q}_K(a)$ as defined over k and with coefficients $\mathbb{Q}[\mathcal{G}_K]$. Our aim in this section is to derive consequences of Theorem 8.16 in the setting of such motives.

10.1. Main conjectures of higher-rank Iwasawa theory

With K/k as above, we write $V(K) \subseteq \Pi_k^\infty$ for the subset of Π_K^∞ comprising places that split in K. For a fixed subset V of Π_k^∞ we then write Ω^V for the collection of K with $V(K) = V$.
Fix a labelling $\Pi_k^\infty := \{v_1, \dots, v_n\}$ and, for every $i \in \{1, \dots, n\}$, fix an extension $w_{k^{\mathrm{c}},i}$ of v_i to k^{c} and write $w_{K,i}$ for the restriction of $w_{k^{\mathrm{c}},i}$ to K. Then $b_\bullet = b_{K,\bullet} := (w_{K,i} : v_i \in V(K))$ is an ordered basis of the free $\mathbb{Z}_p[\mathcal{G}_K]$-module $Y_{V(K)}(\mathbb{Z}_p(1)_{K/k})$.
For any finite subset S of Π_k that contains $\Pi_k^\infty \cup \Pi_k^p$ and finite subset Σ of $\Pi_k \setminus S(K)$, one can use the values at 0 of the $|V|$-th derivatives of Dirichlet L-series over k to define a 'Rubin–Stark element' (depending on $b_\bullet$)

$$\varepsilon^V_{K/k,S(K),\Sigma} \in \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge\nolimits^{|V|}_{\mathbb{Z}[\mathcal{G}_K]} \mathcal{O}^\times_{K,S(K)}$$

(for details see [22, § 5.1]). After fixing an isomorphism $\mathbb{C} \cong \mathbb{C}_p$, we can regard $\varepsilon^V_{K/k,S(K),\Sigma}$ as an element of $\mathbb{C}_p \otimes_{\mathbb{Z}_p} \bigwedge^{|V|}_{\mathbb{Z}_p[\mathcal{G}_K]} H^1(\mathcal{O}_{K,S(K)}, \mathbb{Z}_p(1))$ and the 'p-component' of the Rubin–Stark conjecture (from [91]) predicts that if $H^1_\Sigma(\mathcal{O}_{K,S(K)},\mathbb{Z}_p(1))$ is $\mathbb{Z}_p$-torsion free (or, as is equivalent by Lemma 8.19, the group $H^0_\Sigma(\mathcal{O}_{K,S(K)},(\mathbb{Q}_p/\mathbb{Z}_p)(1))$ vanishes), then one has

$$\varepsilon^V_{K/k,S(K),\Sigma}\in\bigcap\nolimits^{|V|}_{\mathbb{Z}_p[\mathcal{G}_K]}H^1_\Sigma(\mathcal{O}_{K,S(K)},\mathbb{Z}_p(1)).$$

Assuming this conjecture, [91, Prop. 6.1] then shows that we obtain an Euler system

$$\varepsilon_k\coloneqq(\varepsilon^V_{K/k,S(K),\Sigma})_{K\in\Omega^V}\in\mathrm{ES}^r_{\Sigma,S}(\mathbb{Z}_p(1))$$

with $r\coloneqq|V|$.

Let k_∞ be a $\mathbb{Z}_p$-power extension of k in which no finite place splits completely, and set $K_\infty\coloneqq K\cdot k_\infty$. Fix a splitting $\mathcal{G}_{K_\infty}\cong\Delta_K\times\Gamma_K$ with $\Gamma_K\cong\mathbb{Z}_p^n$ for some $n>0$ and Δ_K a finite abelian group, and a direct product decomposition $\Delta_K=\nabla_K\times\Box_K$ as in (8.15). Setting $L\coloneqq K_\infty^{\langle\Gamma_K,\Box_K\rangle}$ and $F\coloneqq K^{\langle\Gamma_K,\nabla_K\rangle}$, we then also have a decomposition $K_\infty^{\Gamma_K}=L\cdot F$. Fix a character $\chi\colon\nabla_K\to\overline{\mathbb{Q}_p}^\times$ and choose an unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$ that contains the values of χ. We consider the representations

$$T_\chi\coloneqq\mathcal{O}(1)(\chi)\quad\text{ and }\quad\mathcal{T}_\chi\coloneqq\Lambda_F(1)(\chi).$$

Writing $\Lambda_K\coloneqq\mathcal{O}[\![\mathcal{G}_{K_\infty}]\!]$, we then have the 'projection map'

$$\Theta_{K_\infty/k,S(K),b_\bullet}\colon\mathrm{Det}_{\Lambda_K}(C^\bullet_{k,\Sigma}(\mathcal{T}_\chi))\to\bigcap\nolimits^r_{\Lambda_K}H^1_\Sigma(\mathcal{O}_{k,S(K)},\mathcal{T}_\chi)$$

from (8.14). Since ε_k belongs to $\mathrm{ES}^r_{\Sigma,S}(\mathbb{Z}_p(1))$, we obtain an element

$$\begin{aligned}\varepsilon^\chi_{K_\infty,\Sigma}\coloneqq(e_\chi\varepsilon^V_{E/k,S(K),\Sigma})_{E\subseteq K_\infty}&\in\varprojlim\nolimits_{E\subseteq K_\infty}\bigcap\nolimits^r_{\mathcal{O}[\mathcal{G}_E]}H^1_\Sigma(\mathcal{O}_{K,S(K)},T_\chi)\\&\cong\bigcap\nolimits^r_{\Lambda_F}H^1_\Sigma(\mathcal{O}_{K,S(K)},\mathcal{T}_\chi).\end{aligned}$$

We can now state the equivariant Iwasawa Main Conjecture for T_χ.

(10.1) Conjecture. *There is a Λ_F-basis $\mathfrak{z}^\chi_{F_\infty,S(K),\Sigma}$ of $\mathrm{Det}_{\Lambda_F}(C_{S(K),\Sigma}(\mathcal{T}_\chi))$ such that*

$$\varepsilon^\chi_{K_\infty,\Sigma}=\Theta_{K_\infty/k,S(K),b_\bullet}(\mathfrak{z}^\chi_{K_\infty,S(K),\Sigma}).$$

(10.2) Remark. The validity of Conjecture 10.1 for all χ in $\widehat{\Delta}$ is equivalent to the validity of the 'rank-r component' of the central conjecture that is formulated by Kurihara, Sano and the second author in [23].

For every finite subextension E of K_∞/k, we write $A_{E,S(K),\Sigma}\coloneqq\mathrm{Cl}_{E,S(K),\Sigma}\otimes_\mathbb{Z}\mathbb{Z}_p$ for the 'p-part' of the $S(K)_E$-ray class group mod Σ_E of E, and then set $A_{K_\infty,S(K),\Sigma}\coloneqq\varprojlim_E A_{K,S(K),\Sigma}$, where the limit is taken with respect to norm maps.

To state our main result towards Conjecture 10.1, we write $\omega_p\colon G_k\to\mu_{p-1}\subseteq\mathbb{Z}_p^\times$ for the p-adic Teichmüller character of k. We remark that in certain situations one can even deduce the validity of the relevant case of Kato's conjecture 8.1 from this result, and we will discuss two such examples in § 10.2 and § 10.3.

(10.3) Theorem. *Assume $p>3$, that $H^1_\Sigma(\mathcal{O}_{K,S(K)},\mathbb{Z}_p(1))$ is $\mathbb{Z}_p$-torsion free, that the p-component of the rank-r Rubin–Stark conjecture holds for all finite abelian extensions of k, and that $\chi\notin\{\mathbf{1},\omega_p\}$. Then one has*

$$\mathrm{Fitt}^0_{\Lambda_F}\left(\mathrm{Ext}^1_{\Lambda_F}\left((\bigcap\nolimits^r_{\Lambda_F}H^1_\Sigma(\mathcal{O}_{K,S(K)},\mathcal{T}_\chi))/(\Lambda_F\varepsilon^\chi_{K_\infty,\Sigma}),\Lambda_F\right)\right)\subseteq\mathrm{Fitt}^0_{\Lambda_F}\left(A^\chi_{K_\infty,S(K),\Sigma}\right)^{**}$$

and

$$\mathrm{Fitt}^0_{\Lambda_F}(Y_{F_\infty,\Pi^p_k}(\mathcal{T}_\chi))^{**}\cdot\varepsilon^\chi_{K_\infty,\Sigma}\subseteq\Theta_{K_\infty/k,S(K),b_\bullet}(\mathrm{Det}_{\Lambda_F}(C_{S(K),\Sigma}(\mathcal{T}_\chi))).$$

Proof. To prove this result we need to verify Hypotheses 8.11 for (T_χ,k_∞,r). Firstly, parts (i) and (ii) of Hypothesis 8.11 are clearly satisfied because $\overline{T_\chi}$ is one-dimensional (in particular, we may take τ to be trivial in (ii)). Furthermore, it is proved in [25, Lem. 5.4] that the condition

$\chi \notin \{\mathbf{1}, \omega_p\}$ implies the vanishing of both $H^1(l_\infty(T_\chi)/k, \overline{T_\chi}^*(1))$ and $H^1(l_\infty(T_\chi)/k, \overline{T_\chi})$. This shows that condition (iii) in Hypothesis 8.11 is valid, and that condition (v) is always valid if $|\Sigma| \leq 1$. Since Conjecture 10.1 is independent of Σ (cf. the argument of [22, Prop. 3.4]) and $H^1_\Sigma(\mathcal{O}_{K,S(K)}, \mathbb{Z}_p(1))$ is automatically $\mathbb{Z}_p$-torsion free if $\Sigma \neq \varnothing$, we may reduce to the case that $|\Sigma| \leq 1$ and thereby Hypothesis 8.11 (v) is valid. Finally, the conditions of Hypothesis 8.11 (ii*) and (iv) are satisfied trivially since we assume $p > 3$.
Having verified the validity of condition (c) in Theorem 8.16, we next note that condition (b) is valid by the assumption that $H^1_\Sigma(\mathcal{O}_{K,S(K)}, \mathbb{Z}_p(1))$ is $\mathbb{Z}_p$-torsion free. To also verify condition (a) in Theorem 8.16, we take $S = S_0$ so that $S_{\mathrm{ram}}(\mathbb{Z}_p(1)) \setminus S_0$ is empty and condition (a) therefore vacuous.
Note that the idempotent ϵ_K from § 8.2 (when taking $Y = Y_{V(K)}(\mathcal{T}_\chi)$) is equal to $\epsilon_{K,V} \coloneqq \prod_{v \in (S_\infty \setminus V)} (1 - e_{\mathcal{G}_{K,v}}) \cdot \prod_{v \in V} e_{\mathcal{G}_{K,v}}$, and hence acts trivially on $\varepsilon_{K_\infty, \Sigma}$. In light of this observation, we now deduce from Theorem 8.16 (iii) with $T \coloneqq \mathbb{Z}_p(1)$ an inclusion

$$\begin{aligned} &\mathrm{Fitt}^0_{\Lambda_F}\Big(\mathrm{Ext}^1_{\Lambda_F}\big(\big(\textstyle\bigcap^r_{\Lambda_F} H^1_\Sigma(\mathcal{O}_{K,S(K)}, \mathcal{T}_\chi)\big)/(\Lambda_F \varepsilon^\chi_{K_\infty,\Sigma}), \Lambda_F\big)\Big) \\ &= \big\{ f(\varepsilon^\chi_{K_\infty,\Sigma}) \mid f \in \textstyle\bigwedge^r_{\Lambda_F} H^1(\mathcal{O}_{K,S(K)}, \mathcal{T}_\chi)^* \big\} \\ &\subseteq \mathrm{Fitt}^0_{\Lambda_F}\big(H^1_{\mathcal{F}_{\mathrm{rel},\Sigma}}(k, \mathcal{T}_\chi^\vee(1))^\vee\big)^{**} \\ &= \mathrm{Fitt}^0_{\Lambda_F}\big(A^\chi_{K_\infty, S(K), \Sigma}\big)^{**}. \end{aligned}$$

(Here the first equality is by [10, Lem. A.10] and the final equality by [92, Prop. 1.6.2].) This proves the first claim in 10.3, and the second claim follows from Theorem 8.16 (i) upon noting that $\mathrm{Fitt}^r_{\Lambda_K}(Y_{\Pi^p_k \cup \Pi^\infty_k}(\mathcal{T}_\chi)) = \epsilon_{K,V}\, \mathrm{Fitt}^0_{\Lambda_K}(Y_{\Pi^p_k}(\mathcal{T}_\chi))$ and $\epsilon_{K,V}$ acts as the identity on $\varepsilon_{K_\infty,V}$. □

10.2. Consequences over imaginary quadratic fields

Throughout this subsection, we assume k is imaginary quadratic and p is odd. We shall first show that if $p > 3$, then Theorem 10.3 implies the validity of Kato's conjecture for $(\mathbb{Q}_K(0), \mathbb{Z}_p[\mathcal{G}_K])$ for every finite abelian extensions K of k. We shall then explain how this result can be combined with the general approach of § 8.3.1 to derive the validity of Kato's Conjecture in other cases. In this way, we realise the strategy discussed by Kato [58, Ch. I, § 3.3] (where the case of motives of the form $\mathbb{Q}(0)_F$ is referred to as 'the universal case') and in a more general context by Huber and Kings [49]. We recall that this approach has already been used extensively in the literature (for more detailed discussion see, for example, [11] and the references therein).

10.2.1. Kato's conjecture for Tate motives

The following consequence of Theorem 10.3 extends the main result of Bley [4].

(10.4) Theorem. *If $p > 3$, then* $\mathrm{TNC}(\mathbb{Q}_K(0), \mathbb{Z}_p[\mathcal{G}_K])$ *is valid for every finite abelian extension K of k.*

Proof. The Conjecture $\mathrm{TNC}(\mathbb{Q}_K(0), \mathbb{Z}_p[\mathcal{G}_K])$ decomposes into the collection of corresponding conjectures for $(h^0(\mathrm{Spec}(L_\chi F)), \mathcal{O}(\chi)[P])$ as χ ranging over the characters of Δ. If $\chi \in \{\mathbf{1}, \omega_p\}$, then the validity of the latter conjecture was proven by Hofer and the first author in [13, Th. B]. (Note that the vanishing of the classical μ-invariant for F, which the result of [13, Th. B] is conditional on, is known as a consequence of the Ferrero–Washington theorem and the fact that $[F : k]$ is a power of p, cf. [13, Prop. 6.7 (b)].) In the following we may therefore assume that $\chi \notin \{\mathbf{1}, \omega_p\}$. We similarly may assume that p does not split in k since otherwise the result of [13, Th. B] applies again.
In the remaining case, then, the second claim of Theorem 10.3 (b) can be improved to give the full Conjecture 10.1. To explain this, we first recall that the rank-1 Rubin–Stark conjecture

holds for all finite abelian extensions of k since the relevant Rubin–Stark elements admit a description in terms of elliptic units (cf. [104, Ch. IV, Prop. 3.9]).
Further, taking k_∞ to be the full $\mathbb{Z}_p^2$-extension of k, the module Y_{K_∞,Π_k^p} becomes pseudo-null over Λ_K (cf. [10, Lem. 6.6 (c)]). A standard argument via localising at height-one primes of Λ_K therefore removes the factor $\mathrm{Fitt}^0_{\Lambda_K}(Y_{K_\infty,\Pi_k^p})$ from the second claim in Theorem 10.3 and the resulting inclusion must be an equality by the analytic class number formula (see [10, Prop. 6.4 (b) (ii)] for details). Having established the equivariant Iwasawa main conjecture (Conjecture 10.1) in this way, the validity of $(h^0(\mathrm{Spec}(L_\chi F)), \mathcal{O}(\chi)[P])$ then follows from the descent formalism developed in [23] combined with [13, Th. A] (see the proof of [13, Th. 6.9] for details). □

The following consequence of Theorem 10.4 strengthens the main result of Johnson-Leung in [52] and itself has a variety of interesting consequences, including the verification of the Quillen–Lichtenbaum Conjecture in a new family of cases (for details of which, see Remark 10.8 below).

(10.5) Corollary. $\mathrm{TNC}(\mathbb{Q}_K(1-j), \mathbb{Z}_p[\mathcal{G}_K])$ *is valid for every finite abelian extension K of k and every integer j with $j > 1$.*

Proof. Throughout this argument we fix $j > 1$ and, as a first step, we explicate the conjecture $\mathrm{TNC}(\mathbb{Q}_K(1-j), \mathbb{Z}_p[\mathcal{G}_K])$. To do this, we fix a finite set $S \subseteq \Pi_k$ containing $\Pi_k^\infty \cup \Pi_k^p \cup S_{\mathrm{ram}}(K/k)$ and recall that the 'Chern class character' map

$$\mathrm{ch}_{K,j} \colon K_{2j-1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1-j))$$

is an isomorphism if p is odd. (This is a consequence of the validity of the Bloch–Kato conjecture that follows from work of Voevodsky and Rost, and completed by Weibel in [112].) Next we fix an embedding $\iota_0 \colon k \hookrightarrow \mathbb{C}$ and recall the 'Beilinson regulator' map (as defined, for example, in [16, § 10.3])

$$\rho_{\mathrm{Bei},K,j} \colon K_{2j-1}(\mathcal{O}_K) \to \mathbb{R} \otimes_{\mathbb{Z}} (K \otimes_k H^0((\mathrm{Spec}\, k)^{\iota_0}(\mathbb{C}), \mathbb{Q}(-j))).$$

Fix a $\mathbb{Z}[\mathcal{G}_K]$-basis η of $K \otimes_k H^0((\mathrm{Spec}\, k)^{\iota_0}(\mathbb{C}), \mathbb{Z}(-j)))$, which amounts to fixing an embedding $\iota \colon \mathbb{Q}^c \hookrightarrow \mathbb{C}$ that extends ι_0, and write $\delta(\eta)$ for the image of η under the comparison isomorphism

$$\mathbb{Z}_p \otimes (K \otimes_k H^0((\mathrm{Spec}\, k)^{\iota_0}(\mathbb{C}), \mathbb{Z}_p(-j)))) \cong \mathcal{Y}_K(\mathbb{Z}_p(1-j)) := \bigoplus\nolimits_{w \in \Pi_K^\infty} H^0(K_w, \mathbb{Z}_p(-j)).$$

Then, since $H^2(\mathcal{O}_{K,S(K)}, \mathbb{Z}_p(1-j))$ is finite (by Soulé [82, Th. 10.3.27]), the construction of (8.6) specialises to the composite map

$$\begin{aligned}\Theta_{K,S,j} \colon \mathrm{Det}_{\mathbb{Q}_p[\mathcal{G}_K]}(C_S(\mathbb{Q}_p(1-j)_{K/k})) &\xrightarrow{\simeq} H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(1-j)) \otimes_{\mathbb{Z}_p[\mathcal{G}_K]} \mathcal{Y}_K(\mathbb{Z}_p(1-j)) \\ &\xrightarrow{\simeq} H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(1-j)),\end{aligned}$$

where the first arrow is the natural 'passage-to-cohomology' map and the second arrow is induced by sending $\delta(\eta) \mapsto 1$. Now, the assumption $j > 1$ implies that, for any finite set $\Sigma \subseteq \Pi_k$ with $\Sigma \cap S(K) = \varnothing$, one has that $\delta_{K,\Sigma}(j) := \prod_{v \in \Sigma}(1 - \mathrm{N}v^{1-j}\mathrm{Frob}_v^{-1})$ is a nonzero divisor in $\mathbb{Z}_p[\mathcal{G}_K]$. It follows that the map $C_{S,\Sigma}(\mathbb{Q}_p(1-j)) \to C_S(\mathbb{Q}_p(1-j))$ is an isomorphism in $D(\mathbb{Q}_p[\mathcal{G}_K])$ and this allows us to regard $\mathrm{Det}_{\mathbb{Z}_p[\mathcal{G}_K]}(C_{S,\Sigma}(\mathbb{Z}_p(1-j)))$ as a submodule of the domain of the map $\Theta_{K,S,j}$. Moreover, the proof of [22, Prop. 3.4] shows that the statement of $\mathrm{TNC}(\mathbb{Q}_K(1-j), \mathbb{Z}_p[\mathcal{G}_K])$ is equivalent to asserting the existence of a $\mathbb{Z}_p[\mathcal{G}_K]$-basis $\mathfrak{z}_{K,\Sigma}(j)$ of $\mathrm{Det}_{\mathbb{Z}_p[\mathcal{G}_K]}(C_{S,\Sigma}(\mathbb{Z}_p(1-j)_{K/k}))$ with both of the following properties

$$\begin{cases}(\mathrm{ch}_{K,j} \circ \Theta_{K,S,j})(\mathfrak{z}_{K,\Sigma}(j)) \in \mathbb{Q} \otimes_{\mathbb{Z}} K_{1-2j}(\mathcal{O}_K);\\ (\rho_{\mathrm{Bei},K,j} \circ \mathrm{ch}_{K,j}^{-1} \circ \Theta_{K,S,j})(\mathfrak{z}_{K,\Sigma}(j)) = \delta_{K,\Sigma}(j) \cdot (\sum_{\chi \in \widehat{\mathcal{G}_K}} L'_S(\chi^{-1}, 1-j)e_\chi) \cdot \eta.\end{cases} \tag{10.6}$$

To construct such a basis, we denote by $\mathfrak{m}$ the conductor of K, and write $w_{\mathfrak{m}}$ for the number of roots of unity in k that are congruent to 1 mod $\mathfrak{m}$ (so $w_{\mathfrak{m}} \mid 12$). Writing $\xi_{\mathfrak{m}}(j) \in K_{1-2j}(\mathcal{O}_K)$ for the element constructed by Johnson-Leung in [52, Th. 3.3], we then define

$$c_K(j) := \mathrm{cores}_{K(p^n\mathfrak{m})/K}(w_{\mathfrak{m}}^{-1} \otimes \xi_{\mathfrak{m}}(j)) \quad \in \mathbb{Z}[1/w_{\mathfrak{m}}] \otimes_{\mathbb{Z}} K_{2j-1}(\mathcal{O}_K)$$

with n big enough such that $K(\mathfrak{p}^n\mathfrak{m})/k$ is ramified at all $\mathfrak{p} \mid p$. Results of Deninger [31], as adapted by Johnson-Leung in [52, Th. 3.3], then show that one has

$$\rho_{\mathrm{Bo},K,j}(c_K(j)) = (-1)^{1+j} \cdot \frac{(2\mathrm{N}\mathfrak{m})^{-(1+j)}}{(-2j)!} \cdot \Big(\sum_{\chi\in\widehat{\mathcal{G}_K}} L'_S(\chi^{-1}, 1-j)e_\chi \Big) \cdot \eta$$

with $S := \Pi_k^\infty \cup \Pi_k^p \cup S_{\mathrm{ram}}(K/k)$ and $\rho_{\mathrm{Bo},K,j}$ the Borel regulator map.

We now write K_∞ for the cyclotomic $\mathbb{Z}_p$-extension of K and set $\Lambda := \mathbb{Z}_p[\![\mathcal{G}_{K_\infty}]\!]$. Using our fixed choice of embedding $\iota\colon \mathbb{Q}^c \hookrightarrow \mathbb{C}$ we can define a basis of $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ as $\zeta := (\iota^{-1}(e^{2\pi i/p^n}))_n$. Choose a prime ideal $\mathfrak{a} \nmid 6p\mathfrak{m}$ and set $\Sigma := \{\mathfrak{a}\}$. We then obtain a map

$$\mathrm{Tw}_j\colon H^1_\Sigma(\mathcal{O}_{k,S}, \Lambda(1)) \xrightarrow{\otimes\zeta^{\otimes -j}} H^1_\Sigma(\mathcal{O}_{k,S}, \Lambda(1)) \otimes_\Lambda \mathbb{Z}_p(-j)_{K/k} \cong H^1_\Sigma(\mathcal{O}_{K,S}, \mathbb{Z}_p(1-j)),$$

where the isomorphism is induced by Proposition 3.45 (d).

Write $\varepsilon_{K_\infty,\Sigma} := (\varepsilon^V_{F/k,S,\Sigma})_{F\subseteq K_\infty}$ for the family of Rubin–Stark elements with $V = \Pi_k^\infty$. Since Rubin–Stark elements in this context admit a description in terms of elliptic units (cf. [13, Ex. 2.3 (c)]), results of Kings [61] show (cf. [52, Th. 3.6]) that

$$(-2j)! \cdot \delta_{K,\Sigma}(j) \cdot \mathrm{ch}_{K,j}(c_K(j)) = \pm \mathrm{N}\mathfrak{m}^{-(1+j)} \cdot \mathrm{Tw}_j(\varepsilon_{K_\infty,\Sigma}),$$

from which we conclude (because $\rho_{\mathrm{Bo},K,j} = 2 \cdot \rho_{\mathrm{Bei},K,j}$ as proved in [16]) that

$$(\rho_{\mathrm{Bei},K,j} \circ \mathrm{ch}_{K,j}^{-1})(\mathrm{Tw}_j(\varepsilon_{K_\infty,\{\mathfrak{a}\}})) = \pm 2^{-j} \cdot \delta_{K,\Sigma}(j) \cdot \big(\textstyle\sum_{\chi\in\widehat{\mathcal{G}_K}} L'_S(\chi^{-1}, 1-j)e_\chi\big) \cdot \eta. \qquad (10.7)$$

By Theorem 10.4, there exists a Λ-basis $\mathfrak{z}_{K,\Sigma}(0)$ of $\mathrm{Det}_\Lambda(C_{S,\Sigma}(\Lambda(1)))$ such that $\Theta_{K,S,1}(\mathfrak{z}_{K,\Sigma}(0)) = \varepsilon_{K_\infty,\Sigma}$. Now, from Lemma 8.17 (i) (a) we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Det}_\Lambda(C_{S,\Sigma}(\Lambda(1))) & \xrightarrow{\Theta_{K,S,1}} & H^1_\Sigma(\mathcal{O}_{k,S}, \Lambda(1)) \\ \Big\downarrow{\scriptstyle \mathrm{Tw}_j^{\det}} & & \Big\downarrow{\scriptstyle \mathrm{Tw_j}} \\ \mathrm{Det}_{\mathbb{Z}_p[\mathcal{G}_K]}(C_{S,\Sigma}(\mathbb{Z}_p(1-j)_{K/k})) & \xrightarrow{\Theta_{K,S,j}} & H^1_\Sigma(\mathcal{O}_{K,S}, \mathbb{Z}_p(1-j)) \end{array}$$

and so we see from (10.7) that $\mathfrak{z}_{K,\Sigma}(j) := \mp 2^j \cdot \mathrm{Tw}_j^{\det}(\mathfrak{z}_{K,\Sigma}(0))$ has both of the properties in (10.6). In addition, since p is odd, $\mathfrak{z}_{K,\Sigma}(j)$ is also a $\mathbb{Z}_p[\mathcal{G}_K]$-basis and so this concludes the proof of the claimed result. □

(10.8) Remark. Corollary 10.5 has a variety of explicit consequences, including the following.

(i) The validity of $\mathrm{TNC}(\mathbb{Q}_K(1-j), \mathbb{Z}_p[\mathcal{G}_K])$ implies that of $\mathrm{TNC}(\mathbb{Q}_K(1-j), \mathbb{Z}_p)$. Hence, upon combining Theorem 10.5 with the result of [21, Th. 2.3], one deduces an equality

$$\mathrm{ord}_p\Big(\frac{\zeta_K^*(1-j)}{R_K(1-j)}\Big) = \mathrm{ord}_p\Big(\frac{|K_{2j-2}(\mathcal{O}_K)|}{|K_{2j-1}(\mathcal{O}_K)_{\mathrm{tor}}|}\Big).$$

Here $\zeta_K^*(1-j)$ denotes the leading term at $s = 1-j$ of the Dedekind ζ-function of K and $R_K(1-j)$ the Borel regulator of K at $1-j$ (as recalled explicitly in [21, Th. 2.1 (iv)]). In particular the displayed equality verifies the Quillen–Lichtenbaum Conjecture (as formulated originally in [69]) in a new family of cases.

(ii) It follows directly from Corollary 10.5 that the main result of El Boukhari in [37] is unconditionally valid for all $p > 3$. From the results obtained in loc. cit. one can therefore immediately derive several concrete consequences of Theorem 10.5 (the details of which we leave to the reader).

(10.9) Remark. While it does not affect the validity of Theorem 10.5 (since p is odd), compatibility requirements on $\mathrm{TNC}(\mathbb{Q}_K(1-j), \mathbb{Z}_2[\mathcal{G}_K])$ suggest that there should not be a factor 2^{-j} in the formula (10.7).

10.2.2. Kato's conjecture for elliptic curves with complex multiplication

We now derive from Theorem 10.4 an equivariant version of the main result of Burungale and Flach in [30]. To do this we fix an elliptic curve E that is defined over a number field F containing k and has complex multiplication by the ring of integers $\mathcal{O}_k$ of k. We assume that $F(E_{\mathrm{tors}})/k$ is abelian. This implies that the Weil restriction $B \coloneqq \mathrm{Res}_k^F(E)$ of E is an abelian variety of dimension $[F:k]$ defined over k with the property that

$$A \coloneqq \mathrm{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Q} \cong L_1 \times \cdots \times L_t,$$

where each L_i is a CM field that contains k and one has $\sum_{i=1}^t [L_i : k] = [F : k]$ (cf. [30, Prop. 3.2]). We also fix an odd prime p and set $\mathcal{O}_p \coloneqq \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_k$ and $\mathcal{A}_p \coloneqq \mathrm{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

(10.10) Lemma. *There is an isomorphism $\mathcal{A}_p \cong \mathcal{O}_p[\mathcal{G}_F]$ of rings and an isomorphism $T_pB \cong \mathrm{Ind}_{G_F}^{G_k}(T_pE)$ of G_k-modules. In particular, $\mathcal{A}_p$ is Gorenstein and T_pB is a free $\mathcal{A}_p$-module of rank one.*

Proof. For every $\sigma \in \mathcal{G}_F$ we denote by E^σ the σ-conjugate of E. Then σ induces an isomorphism of abelian groups $E(\mathbb{Q}^c) \xrightarrow{\simeq} E^\sigma(\mathbb{Q}^c)$. From the decomposition $B(\mathbb{Q}^c) = \prod_{\sigma\in\mathcal{G}_F} E^\sigma(\mathbb{Q}^c)$ we then see that $T_pB = \prod_{\sigma\in\mathcal{G}_F} T_pE^\sigma \cong \mathrm{Ind}_{G_F}^{G_k}(T_pE)$, as claimed (cf. also [79, (a) on p. 178]).
Label the elements of $\mathcal{G}_F$ as $\bar\sigma_1, \dots, \bar\sigma_n$ and, for every i, fix $\sigma_i \in G_k$ that restricts to $\bar\sigma_i$. These choices then define an isomorphism

$$f\colon \mathrm{Ind}_{G_F}^{G_k}(T_pE) = T_pE \otimes_{\mathbb{Z}_p[\![G_F]\!]} \mathbb{Z}_p[\![G_k]\!] \to T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathcal{G}_F], \quad a \otimes \sigma_i h \mapsto (h\cdot a) \otimes \bar\sigma_i$$

of $\mathbb{Z}_p$-modules (here $h \in G_F$). We claim that this isomorphism is compatible with the action of $\mathbb{Z}_p[\![G_k]\!]$ when acting $T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathcal{G}_F]$ via the isomorphism $\mathbb{Z}_p[\![G_k]\!] \cong \mathbb{Z}_p[\![G_F]\!][\mathcal{G}_F]$. To do this, we suppose to be given an element of the form $g_jh' \in G_k$ and compute that

$$f(\sigma_jh' \cdot a \otimes \sigma_ih) = f(a \otimes \sigma_j\sigma_i(\sigma_i^{-1}h'\sigma_ih)) = (\sigma_i^{-1}h'\sigma_ih\cdot a) \otimes \sigma_j\sigma_i = (h'h\cdot a)\otimes \sigma_j\sigma_i,$$

where the last equality uses that σ_i acts trivially (by conjugation) on $\mathrm{Gal}(F(E_{\mathrm{tors}})/F)$ because $F(E_{\mathrm{tors}})/k$ is assumed to be abelian.
Write $\psi\colon G_F \to \mathrm{Aut}(T_pE) \cong \mathcal{O}_p^\times$ for the character induced by the action of G_F on T_pE, then the action of $\mathbb{Z}_p[\![G_F]\!][\mathcal{G}_F]$ on $T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathcal{G}_F]$ factors through the morphism $\mathbb{Z}_p[\![G_k]\!] \to \mathcal{O}_p[\mathcal{G}_F]$ induced by ψ.
To investigate $\mathcal{A}_p$ we use that the known validity of the Tate conjecture for abelian varieties over number fields (by Faltings [38]) implies that we have an isomorphism of rings $\mathcal{A}_p \cong \mathrm{End}_{\mathbb{Z}_p[\![G_k]\!]}(T_pB)$. Writing $\mathcal{R} \subseteq \mathcal{O}_p$ for the $\mathbb{Z}_p$-order generated by the image of ψ, we therefore have that

$$\mathrm{End}_{\mathbb{Z}_p[\![G_k]\!]}(T_pB) = \mathrm{End}_{\mathcal{R}[\mathcal{G}_F]}(T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathcal{G}_F]) = \mathrm{End}_{\mathcal{O}_p[\mathcal{G}_F]}(T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathcal{G}_F]) \cong \mathcal{O}_p[\mathcal{G}_F],$$

where the second equality holds because $\mathcal{R}$ is of finite index in $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_k$ (as the cokernel of ψ is finite) and T_pB is $\mathbb{Z}_p$-torsion free, and the last isomorphism holds because $T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathcal{G}_F]$ is a free $\mathcal{O}_p[\mathcal{G}_F]$-module (cf. also [101, Rk. on p. 502]). □

Before stating our main result, we now quickly review the precise formulation of the 'analytic-rank-zero' component of $\mathrm{TNC}(h^1(B/K)(1), \mathcal{A}_p[\mathrm{Gal}(K/F)])$.
We begin by recalling the $(\mathbb{C} \otimes_{\mathbb{Q}} A)[\mathcal{G}_K]$-valued L-function attached to the motive $M_K \coloneqq h^1(B_K)(1)$. Fix an isogeny $B \to \prod_{i=1}^t B_i$ with each B_i a simple abelian variety defined over k with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_k(B_i) \cong L_i$. Attached to each B_i is then an algebraic Hecke character (the 'Serre–Tate character' of B_i) $\varphi_i\colon \mathbb{A}_k^\times \to L_i^\times$ of infinity type $(-1,0)$, which gives rise, for every $\tau \in \mathrm{Hom}(L_i, \mathbb{C})$, to a Hecke character

$$\varphi_{i,\tau}\colon \mathbb{A}_k^\times/k^\times \to \mathbb{C}^\times$$

(see [101, Th. 10] for details). Setting $S \coloneqq \Pi_k^\infty \cup S_{\mathrm{ram}}(K/k) \cup S_{\mathrm{ram}}(B)$, one then has

$$L_S(M_K, s) = \Big(\sum\nolimits_{\chi\in\widehat{\mathcal{G}_K}} L_S(\overline{\varphi_{i,\tau}}\chi^{-1}, s+1)e_\chi\Big)_\tau \quad \in \bigoplus\nolimits_{\tau\colon L_i\hookrightarrow\mathbb{C}} \mathbb{C}[\mathcal{G}_K] \cong (\mathbb{C}\otimes_{\mathbb{Q}} L_i)[\mathcal{G}_K].$$

To describe the period map, it is convenient to fix an embedding $\iota\colon k \hookrightarrow \mathbb{C}$ so that one has an identification $(\bigoplus_{\tau\colon k\hookrightarrow\mathbb{C}} H^1(B^\tau(\mathbb{C}),\mathbb{Q}))^+ \cong H^1(B^\iota(\mathbb{C}),\mathbb{Q})$. Using the isomorphism $H^1(E^\iota(\mathbb{C}),\mathbb{Q}) \cong H_1(E^\iota(\mathbb{C}),\mathbb{Q})^*$ from Poincaré duality, we may define the period map of B_K as

$$\mathrm{per}_{B,K}\colon K \otimes_k H^0(B,\Omega^1_{B/k}) \to (\mathbb{R}\otimes_\mathbb{Q} K)\otimes_k H^1(B^\iota(\mathbb{C}),\mathbb{Q}), \quad \omega \mapsto \{\gamma \mapsto \int_\gamma \omega\}.$$

Next we note that if $L(B/K,1) \neq 0$, then $B(K)$ and $Ш_{B/K}$ are known to both be finite (cf. [30, Th. 1.2], and this implies that $H^2(\mathcal{O}_{K,S},V_pB)$ vanishes in this case.
Let γ denote a choice of $\mathcal{A} \coloneqq \mathrm{End}_k(B)$-basis of $H^1(B^\iota(\mathbb{C}),\mathbb{Z})$, and write $\delta(\gamma)$ for the image of γ under the comparison isomorphism $\mathbb{Z}_p\otimes_\mathbb{Z} H^1(B^\iota(\mathbb{C}),\mathbb{Z}) \cong H^1(B,\mathbb{Z}_p) \cong (T_pB)^*$. Setting $\mathcal{T} \coloneqq \mathrm{Ind}_{G_K}^{G_k}(T_pB)$, the construction of (8.6) defines a map

$$\begin{aligned}\Theta_{K,S,\delta(\gamma)}\colon \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{Det}_{\mathcal{A}_p[\mathcal{G}_K]}(C_S(\mathcal{T})) &\xrightarrow{\simeq} H^1(\mathcal{O}_{K,S},V_pB)\otimes_{(\mathbb{Q}_p\otimes_\mathbb{Q} A)[\mathcal{G}_K]}(V_pB)^*\\ &\xrightarrow{\simeq} H^1(\mathcal{O}_{K,S},V_pB),\end{aligned}$$

where the first arrow is the natural 'passage-to-cohomology' map and the second arrow is induced by sending $\delta(\gamma)\mapsto 1$. We also define the composite map

$$\begin{aligned}\lambda_{B,K,S}\colon H^1(\mathcal{O}_{K,S},V_pB) &\to \bigoplus\nolimits_{v|p} H^1(K_v,V_pB)\\ &\xrightarrow{\exp^*} D^1_{\mathrm{dR},\mathbb{Q}_p\otimes_\mathbb{Q} K}(V_pB)\\ &\xrightarrow{\simeq} (\mathbb{Q}_p\otimes_\mathbb{Q} K)\otimes_\mathbb{Q} H^0(B,\Omega^1_{B/k})\end{aligned}$$

with the first arrow the natural localisation morphism, the second arrow the dual exponential map of V_pB [57, Ch. II, § 1.2.4], and the last arrow the arrow the comparison isomorphism from p-adic Hodge theory.
We also fix a finite set $\Sigma \subseteq \Pi_k$ that is disjoint from S and define the element

$$\delta_{K,v}(X) \coloneqq \det_{\mathcal{A}_p[\mathcal{G}_K]}(1-\mathrm{Frob}_v^{-1}X \mid T_pB) \in \mathcal{A}_p[X]$$

for every $v\in\Sigma$. Since $H^0(K_w,T_pB)$ vanishes for all $w\in\Sigma_K$, the element $\delta_{K,\Sigma} \coloneqq \prod_{v\in\Sigma}\delta_{K,v}(\mathrm{Frob}_v^{-1})$ is a nonzero divisor in $\mathcal{A}_p[\mathcal{G}_K]$ and, as a consequence, the argument of [22, Prop. 3.4] shows that $\mathrm{TNC}(h^1(B/K)(1),\mathcal{A}_p[\mathrm{Gal}(K/k)])$ is equivalent to the existence of an $\mathcal{A}_p[\mathcal{G}_K]$-basis $\mathfrak{z}_{B/K,\Sigma}$ of $\mathrm{Det}_{\mathcal{A}_p[\mathcal{G}_K]}(C_{S,\Sigma}(\mathcal{T}))$ that has both of the following properties

$$\begin{cases}(\lambda_{B,K,S}\circ\Theta_{K,S,\delta(\gamma)})(\mathfrak{z}_{B/K,\Sigma}) \in K\otimes_k H^0(B,\Omega^1_{B/K});\\ (\mathrm{per}_{B,K}\circ\lambda_{B,K,S}\circ\Theta_{K,S,\delta(\gamma)})(\mathfrak{z}_{B/K,\Sigma}) = \delta_{K,\Sigma}\cdot L_S(M,0)\cdot\gamma.\end{cases} \tag{10.11}$$

We now state the main result of this section.

(10.12) Theorem. *Fix data E/F and B as above and let K be a finite abelian extension of k for which $L(B/K,1)\neq 0$. Then $\mathrm{TNC}(h^1(B/K)(1),\mathcal{A}_p[\mathrm{Gal}(K/k)])$ is valid.*

Proof. This argument is a special case of the general strategy described in [11, Th. 3.22]. Consider the character

$$\rho\colon G_k \to \mathrm{Aut}(\mathcal{T}) \cong \mathcal{A}_p[\mathcal{G}_K]^\times$$

and write $L \coloneqq (k^c)^{\ker\rho}$ for the field cut out by ρ. (This is an abelian extension of k which contains K and a $\mathbb{Z}_p$-extension of k.) We also set $\Lambda \coloneqq \mathbb{Z}_p[\![\mathcal{G}_L]\!]$, take $S\subseteq\Pi_k$ to be a finite set that contains $\Pi_k^\infty\cup\Pi_k^p\cup S_{\mathrm{ram}}(T_pB)\cup S_{\mathrm{ram}}(K/k)$, and let $\Sigma \coloneqq \{\mathfrak{a}\}$ with $\mathfrak{a}$ a prime ideal of k that does not belong to S.
To construct $\mathfrak{z}_{B/K}$, we will use that by Theorem 10.4 we have a Λ-basis $\mathfrak{z}_{\mathbb{G}_m/L,\Sigma}$ of $\mathrm{Det}_\Lambda(C_{S,\Sigma}(\Lambda(1)))$ with the property that the map

$$\mathrm{Det}_\Lambda(C_{S,\Sigma}(\Lambda(1))) \to H^1_\Sigma(\mathcal{O}_{k,S},\Lambda(1))$$

sends $\mathfrak{z}_{\mathbb{G}_m/L,\Sigma}$ to the family of Rubin–Stark elements $\varepsilon_{L,\Sigma} \coloneqq (\varepsilon^{\{\infty\}}_{F/k,S,\Sigma})_{F\subseteq L}$.
Now, the isomorphism $\Lambda(1)\otimes_{\Lambda,\rho^{-1}}\mathcal{A}_p \cong \mathcal{T}^*(1) \cong \mathcal{T}$ of G_k-representations combines with

Proposition 3.45 (iv) to give an isomorphism $C_{S,\Sigma}(\Lambda(1)) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{A}_p \cong C_{S,\Sigma}(\mathcal{T})$, and from this we deduce the existence of a morphism of Λ-modules

$$\mathrm{Tw}^{\mathrm{det}} \colon \mathrm{Det}_{\Lambda}(C_{S,\Sigma}(\Lambda(1))) \to \mathrm{Det}_{\Lambda}(C_{S,\Sigma}(\Lambda(1))) \otimes_{\Lambda} \mathcal{A}_p \cong \mathrm{Det}_{\mathcal{A}_p}(C_{S,\Sigma}(\mathcal{T}))$$

that, by Lemma 8.17 (i) (a), lies in a commutative diagram

$$\begin{array}{ccc} \mathrm{Det}_{\Lambda}(C_{S,\Sigma}(\Lambda(1)) & \longrightarrow & H^1_{\Sigma}(\mathcal{O}_{k,S}, \Lambda(1)) \\ \downarrow{\scriptstyle \mathrm{Tw}^{\mathrm{det}}} & & \downarrow{\scriptstyle \mathrm{Tw}} \\ \mathrm{Det}_{\mathcal{A}_p}(C_{S,\Sigma}(\mathcal{T})) & \xrightarrow{\Theta_{K,S,\delta(\gamma)}} & H^1_{\Sigma}(\mathcal{O}_{K,S}, V_pB). \end{array}$$

Here we write Tw for the map

$$\begin{aligned} \mathrm{Tw} \colon H^1_{\Sigma}(\mathcal{O}_{k,S}, \Lambda(1)) &\xrightarrow{\gamma} H^1_{\Sigma}(\mathcal{O}_{k,S}, \Lambda(1)) \otimes_{\Lambda} (V_pB)^* \\ &\xrightarrow{\simeq} H^1_{\Sigma}(\mathcal{O}_{K,S}, (V_pB)^*(1)), \end{aligned}$$

where the first arrow is induced by sending $1 \mapsto \delta(\gamma)$ and the second is the isomorphism that arises from Proposition 3.45 (iv).

By Lemma 8.17 (i) (b) the element $\mathfrak{z}_{B/K,\Sigma} \coloneqq \mathrm{Tw}^{\mathrm{det}}(\mathfrak{z}_{\mathbb{G}_m/L,\Sigma})$ is an $\mathcal{A}_p$-basis of $\mathrm{Det}_{\mathcal{A}_p}(C_{S,\Sigma}(\mathcal{T})$, and so it suffices to verify that this element also has both of the properties in (10.11). Given the above commutative diagram, we can therefore verify the latter properties after replacing the element $\Theta_{K,S,\delta(\gamma)}(\mathfrak{z}_{B/K,\Sigma})$ by $\mathrm{Tw}(\varepsilon_{L,\Sigma})$, and to do this we use the reciprocity law of Kato–Wiles. Specifically, after taking account of [30, proof of Prop. 3.3], which compares the spaces $V_{L_i}(\varphi_i)$ and $S(\varphi_i)$ introduced in [60, § 15.8] with $H^1(B^{\iota}, \mathbb{Q})$ and $H^0(B, \Omega^1_{B/K})$, this reciprocity law [60, Prop. 15.9] asserts that $(\lambda_{B,K,S} \circ \mathrm{Tw})(\varepsilon_{L,\Sigma})$ belongs to $K \otimes_{\mathbb{Q}} H^0(B, \Omega^1_{B/k})$ and satisfies

$$\sum\nolimits_{\sigma \in \mathcal{G}_K} \chi(\sigma) \cdot \mathrm{per}_B(\sigma \cdot (\lambda_{B,K,S} \circ \mathrm{Tw})(\varepsilon_{L,\Sigma})) = (\tau \otimes \chi^{-1})(\delta_{K,\Sigma}) \cdot (L_S(\overline{\varphi}_{\tau} \cdot \chi, 1))_{\tau} \otimes \gamma.$$

Here τ ranges over the elements of $\mathrm{Hom}(L_i, \mathbb{C})$ with $\tau_{|k} = \iota$ so that $(L_S(\overline{\varphi}_{\tau} \cdot \chi, 1))_{\tau}$ is an element of $L_i \otimes_{\mathbb{Q}} \mathbb{R}$. The required result then follows directly from the fact that

$$L_S(M, 0) = \sum\nolimits_{i=1}^{t} \sum\nolimits_{\tau} \sum\nolimits_{\chi \in \widehat{\mathcal{G}_K}} L_S(\overline{\varphi}_{\tau}\chi, 1) e_{\chi^{-1}} \in (\mathbb{R} \otimes_{\mathbb{Q}} A)[\mathcal{G}_K]. \qquad \square$$

(10.13) Remark. (i) If $L(B/K, 1) = 0$, then the argument of Theorem 10.12 proves the 'analytic-rank-zero' component of $\mathrm{TNC}(h^1(B/K)(1), \mathcal{A}_p[\mathrm{Gal}(K/F)])$.

(ii) Since $T_pB \cong \mathrm{Ind}^{G_k}_{G_F}(T_pE)$ by Lemma 10.10, standard functoriality properties of Kato's conjecture combine with Theorem 10.12 to imply the validity of the Birch–Swinnerton-Dyer Conjecture for E/F. This recovers the main result of Burungale and Flach in [30], and is indeed very close in spirit to the strategy employed in loc. cit.

10.3. More general cases

Theorem 10.4, and its method of proof, have consequences well beyond the case of abelian extensions of imaginary fields. As an example, we offer the following result which is derived by incorporating the general approach developed by Daoud, Seo and the present authors in [10]. For an abelian extension K of k, we set $S^*(K) \coloneqq \Pi_k^{\infty} \cup S_{\mathrm{ram}}(K/k)$.

(10.14) Theorem. *Assume $p > 3$ is a prime number that is inert in k and does not divide the class number of k. Fix a non-empty subset V of Π_k^{∞} and assume the Rubin–Stark conjecture to be valid for all data $(K/k, S^*(K), \Sigma_K, V)$ with K a finite abelian extension of k and Σ_K a finite subset of $\Pi_k \setminus S^*(K)$. Then $\mathrm{TNC}(\mathbb{Q}_K(0), \epsilon_{K,V}\mathbb{Z}_p[\mathcal{G}_K])$ is valid for every finite abelian extension K of k with $\mu_K[p] = (0)$.*

Proof. We write Ω^V for the collection of all finite abelian extensions K of k such that $V(K) = V$, and $\omega_p \colon G_k \to \mathbb{Z}_p^{\times}$ for the p-adic Teichmüller character of k. Then the proof of [10,

Th. 6.1 (a) (i)] shows that $\mathrm{TNC}(\mathbb{Q}_K(0),(1-e_{\omega_p})\epsilon_{K,V}\mathbb{Z}_p[\mathcal{G}_K])$ is valid for every finite abelian extension K of k if Conjecture 10.1 is valid for all $K\in\Omega^V$ and characters $\chi\in\widehat{\nabla_K}\setminus\{\omega_p\}$ with k_∞ taken to be the cyclotomic $\mathbb{Z}_p$-extension of k. (Note that, since our formulation of Conjecture 10.1 is explicitly in terms of the determinant functor and perfect complexes, we do not need to assume condition (ii) of [10, Th. 6.1 (a) (i)].) In particular, in this case, $\mathrm{TNC}(\mathbb{Q}_K(0),\epsilon_{K,V}\mathbb{Z}_p[\mathcal{G}_K])$ is valid for every finite abelian extension K of k with $\mu_K[p]=(0)$.
To verify the validity of Conjecture 10.1 in the required cases, we then proceed similarly as in Theorem 10.4. To be precise, we claim first it suffices to prove, for all $K\in\Omega^V$, finite sets $\Sigma_K\subseteq\Pi_k\setminus(S^*(K)\cup\Pi_k^p)$, and characters $\chi\in\widehat{\nabla_K}\setminus\{\omega_p\}$, that there is an inclusion

$$\mathrm{Fitt}^0_{\Lambda_F}(Y_{F_\infty,\Pi_k^p}(\mathcal{T}_\chi))^{**}\cdot\varepsilon^\chi_{K_\infty,\Sigma_K}\subseteq\Theta_{K_\infty/k,S(K),b_\bullet}(\mathrm{Det}_{\Lambda_F}(C_{S(K),\Sigma_K}(\mathcal{T}_\chi))),\tag{10.15}$$

with $\mathcal{T}_\chi:=\Lambda_F(1)(\chi)$.
To show Conjecture 10.1 is indeed implied by these inclusions, we may work locally at a fixed height-one prime $\mathfrak{p}$ of Λ_F. If $p\in\mathfrak{p}$, then one has $\mathrm{Fitt}^0_{\Lambda_F}(Y_{F_\infty,\Pi_k^p}(\mathcal{T}_\chi))_\mathfrak{p}=\Lambda_{F,\mathfrak{p}}$ by Lemma 8.22, and so the above inclusion implies

$$\varepsilon^\chi_{K_\infty,\Sigma_K}\in\Theta_{K_\infty/k,S(K),b_\bullet}(\mathrm{Det}_{\Lambda_F}(C_{S(K),\Sigma_K}(\mathcal{T}_\chi)))_\mathfrak{p}.\tag{10.16}$$

To prove the same containment also holds if $p\notin\mathfrak{p}$, we recall the module $A_{K_\infty,S(K),\Sigma_K}$ defined just before the statement of Theorem 10.3, and then note that the argument of Theorem 8.16 (iii) shows that (10.15) implies an inclusion

$$\mathrm{im}(\varepsilon^\chi_{K_\infty,\Sigma_K})_\mathfrak{p}\subseteq\mathrm{Fitt}^0_{\Lambda_F}(A^\chi_{K_\infty,S(K),\Sigma_K})_\mathfrak{p}.\tag{10.17}$$

Next, we note that, since the given assumptions imply $|\Pi_k^p|=1$, the 'Gross–Kuz'min conjecture' for K is known to be valid by a result of Maksoud [71, Th. 4.3.2 (b)]. As a consequence, we may combine the results of [10, Lem. 6.6 (b) and (d)] with (10.17) to deduce that

$$\mathrm{im}(\varepsilon^\chi_{K_\infty,\Sigma_K})_\mathfrak{p}\subseteq\mathrm{Fitt}^0_{\Lambda_F}(A^\chi_{K_\infty,S(K),\Sigma_K})_\mathfrak{p}\cdot\mathrm{Fitt}^0_{\Lambda_F}(X_{F_\infty,\Pi_k^p}(\mathcal{T}_\chi))_\mathfrak{p}.$$

Since this inclusion holds for all $K\in\mathcal{X}$ that are unramified at Σ_K we may then use the argument of [10, Lem. 6.6 (a)] to deduce that

$$\mathrm{im}(\varepsilon^\chi_{K_\infty,\Sigma_K})_\mathfrak{p}\subseteq\mathrm{Fitt}^0_{\Lambda_F}(A^\chi_{K_\infty,S(K),\Sigma_K})_\mathfrak{p}\cdot\mathrm{Fitt}^0_{\Lambda_F}(X_{F_\infty,S(K)_{\mathrm{fin}}}(\mathcal{T}_\chi))_\mathfrak{p}.$$

By [10, Prop. 6.4 (b) (i)] this then implies (10.16) for $\mathfrak{p}$. Having now verified that the latter inclusion is valid for every height-one prime ideal of Λ_F, the analytic class number formula allows us to deduce the validity of Conjecture 10.1 (cf. the argument of [10, Prop. 6.4 (b) (ii)]).
At this stage, it therefore only remains for us to justify the inclusions (10.15). In addition, this inclusion follows directly from Theorem 9.4 in the case $\chi\neq\mathbf{1}$, and so it is enough for us to prove it in the case $\chi=\mathbf{1}$.
To this end, we first recall the well-known fact that the p-adic Iwasawa μ-invariant of k vanishes since $|\Pi_k^p|=1$ and p does not divide the class number of k (see [110, Prop. 13.22]). By a standard argument (cf. [23, Prop. 3.15]), the proof of (10.15) is therefore reduced to showing, for every $\psi\in\widehat{\square_K}$, that there is an inclusion of characteristic ideals

$$\mathrm{char}_{\Lambda_\psi}\big((\textstyle\bigcap^r_{\Lambda_\psi}H^1_{\Sigma_K}(\mathcal{O}_{k,S(K)},\Lambda_\psi(1)(\psi))/(\Lambda_\psi\varepsilon^\psi_{K_\infty,\Sigma_K})\big)\subseteq\mathrm{char}_{\Lambda_\psi}(A^\psi_{K_\infty,S(K),\Sigma_K}).$$

If $\psi\neq\mathbf{1}$, then one can use Proposition 5.28 to directly deduce this from [92, Th. 2.3.3]. The remaining case of $\psi=\mathbf{1}$ (which might not validate the hypothesis Hyp (K_∞/K) in [92] and so has to be considered separately) is trivial because the assumptions that k is a field with only one p-adic place and of class number not divisible by p implies that $A_{k_\infty,S(K)}$ vanishes (cf. [110, Prop. 13.22]). □

(10.18) Remark. Fix a non-empty subset V of Π_k^∞ and a subset $\mathcal{X}$ of Ω^V that satisfies the 'closure hypothesis' of [10, Hyp. 4.5]. Also fix a $\mathbb{Z}_p$-power extension k_∞ of k in which no finite place splits completely, and consider the following conditions.

(i) $p>3$ is a prime number that is unramified in k.

(ii) If $K \in \mathcal{X}$, then the Rubin–Stark conjecture holds for all data $(K/k, S^*(K), \Sigma, V)$ as described in the statement of Theorem 10.14.

(iii) Conjecture 10.1 holds if $K \in \mathcal{X}$ and $\chi \in \{\mathbf{1}, \omega_p\}$.

(iv) One has $\mathrm{im}(\varepsilon^V_{K_\infty,\Sigma}) \subseteq \mathrm{Fitt}^0_{\Lambda_K}(X_{K_\infty,\Pi^p_k})^{**}$ for all $K \in \mathcal{X}$.

(v) The supports of X_{K_∞,Π^p_k} and $A_{K_\infty,S(K)}$ are disjoint.

Then a more careful use of the argument proving Theorem 10.14 shows that, in any situation in which all of the above conditions are satisfied, the conjecture $\mathrm{TNC}(\mathbb{Q}_K(0), \epsilon_{K,V}\mathbb{Z}_p[\mathcal{G}_K])$ is valid for every finite (abelian) extension K of k in $\mathcal{X}$.

(10.19) Example. Suppose k is a complex cubic number field and $V \subseteq \Pi^\infty_k$ is the singleton comprising the complex place of k. In this setting Bergeron–Charollois–García [2] have recently provided strong evidence for the Rubin–Stark conjecture for the data $(K/k, S^*(K), \Sigma, V)$. Assuming the latter conjecture to be valid, there is then a positive-density set of primes p for which Theorem 10.14 can be applied. Indeed, since the normal closure $\widetilde{k}$ of k has Galois group $\mathrm{Gal}(\widetilde{k}/k) \cong S_3$, the Cebotarev density theorem provides us with a positive-density set of primes p such that the alternating group A_3 is generated by the conjugacy class of $\mathrm{Frob}_\mathfrak{p}$ for any prime of $\widetilde{k}$ lying above $\mathfrak{p}$. In particular, any prime in this set that does not divide the class number of k satisfies the assumptions of Theorem 10.14.

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